Harvard-MIT Mathematics Tournament

March 15, 2003

Individual Round: Algebra Subject Test

1. Find the smallest value of x such that $a \geq 14\sqrt{a} - x$ for all nonnegative a.

2. Compute $\frac{\tan^2(20°)-\sin^2(20°)}{\tan^2(20°)\sin^2(20°)}$.

3. Find the smallest n such that $n!$ ends in 290 zeroes.

4. Simplify: $2\sqrt{1.5+\sqrt{2}}-(1.5+\sqrt{2})$.

5. Several positive integers are given, not necessarily all different. Their sum is 2003. Suppose that n_1 of the given numbers are equal to 1, n_2 of them are equal to 2, $\ldots$, n_{2003} of them are equal to 2003. Find the largest possible value of
$$n_2 + 2n_3 + 3n_4 + \cdots + 2002n_{2003}.$$

6. Let $a_1 = 1$, and let $a_n = \lfloor n^3/a_{n-1} \rfloor$ for $n > 1$. Determine the value of a_{999}.

7. Let a, b, c be the three roots of $p(x) = x^3 + x^2 - 333x - 1001$. Find $a^3 + b^3 + c^3$.

8. Find the value of $\frac{1}{3^2+1} + \frac{1}{4^2+2} + \frac{1}{5^2+3} + \cdots$.

9. For how many integers n, for $1 \leq n \leq 1000$, is the number $\frac{1}{2}\binom{2n}{n}$ even?

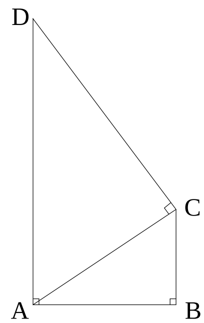
10. Suppose $P(x)$ is a polynomial such that $P(1) = 1$ and
$$\frac{P(2x)}{P(x+1)} = 8 - \frac{56}{x+7}$$
for all real x for which both sides are defined. Find $P(-1)$.

Harvard-MIT Mathematics Tournament

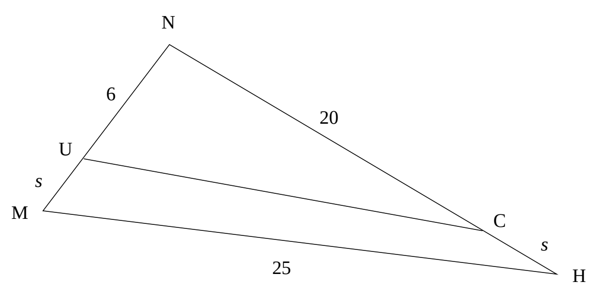
March 15, 2003

Individual Round: Geometry Subject Test

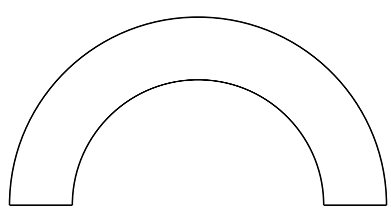
1. AD and BC are both perpendicular to AB, and CD is perpendicular to AC. If $AB = 4$ and $BC = 3$, find CD.



2. As shown, U and C are points on the sides of triangle MNH such that $MU = s$, $UN = 6$, $NC = 20$, $CH = s$, $HM = 25$. If triangle UNC and quadrilateral $MUCH$ have equal areas, what is s?



3. A room is built in the shape of the region between two semicircles with the same center and parallel diameters. The farthest distance between two points with a clear line of sight is 12m. What is the area (in m^2) of the room?



4. Farmer John is inside of an ellipse with reflective sides, given by the equation $x^2/a^2 + y^2/b^2 = 1$, with $a > b > 0$. He is standing at the point $(3, 0)$, and he shines a laser pointer in the y-direciton. The light reflects off the ellipse and proceeds directly toward Farmer Brown, traveling a distance of 10 before reaching him. Farmer John then spins around in a circle; wherever he points the laser, the light reflects off the wall and hits Farmer Brown. What is the ordered pair (a, b)?

5. Consider a 2003-gon inscribed in a circle and a triangulation of it with diagonals intersecting only at vertices. What is the smallest possible number of obtuse triangles in the triangulation?

6. Take a clay sphere of radius 13, and drill a circular hole of radius 5 through its center. Take the remaining "bead" and mold it into a new sphere. What is this sphere's radius?

7. Let $RSTUV$ be a regular pentagon. Construct an equilateral triangle PRS with point P inside the pentagon. Find the measure (in degrees) of angle PTV.

8. Let ABC be an equilateral triangle of side length 2. Let ω be its circumcircle, and let $\omega_A, \omega_B, \omega_C$ be circles congruent to ω centered at each of its vertices. Let R be the set of all points in the plane contained in exactly two of these four circles. What is the area of R?

9. In triangle ABC, $\angle ABC = 50°$ and $\angle ACB = 70°$. Let D be the midpoint of side BC. A circle is tangent to BC at B and is also tangent to segment AD; this circle instersects AB again at P. Another circle is tangent to BC at C and is also tangent to segment AD; this circle intersects AC again at Q. Find $\angle APQ$ (in degrees).

10. Convex quadrilateral $MATH$ is given with $HM/MT = 3/4$, and $\angle ATM = \angle MAT = \angle AHM = 60°$. N is the midpoint of MA, and O is a point on TH such that lines MT, AH, NO are concurrent. Find the ratio HO/OT.

Harvard-MIT Mathematics Tournament

March 15, 2003

Individual Round: Combinatorics Subject Test

1. You have 2003 switches, numbered from 1 to 2003, arranged in a circle. Initially, each switch is either ON or OFF, and all configurations of switches are equally likely. You perform the following operation: for each switch S, if the two switches next to S were initially in the same position, then you set S to ON; otherwise, you set S to OFF. What is the probability that all switches will now be ON?

2. You are given a 10×2 grid of unit squares. Two different squares are adjacent if they share a side. How many ways can one mark exactly nine of the squares so that no two marked squares are adjacent?

3. Daniel and Scott are playing a game where a player wins as soon as he has two points more than his opponent. Both players start at par, and points are earned one at a time. If Daniel has a 60% chance of winning each point, what is the probability that he will win the game?

4. In a certain country, there are 100 senators, each of whom has 4 aides. These senators and aides serve on various committees. A committee may consist either of 5 senators, of 4 senators and 4 aides, or of 2 senators and 12 aides. Every senator serves on 5 committees, and every aide serves on 3 committees. How many committees are there altogether?

5. We wish to color the integers $1, 2, 3, \ldots, 10$ in red, green, and blue, so that no two numbers a and b, with $a - b$ odd, have the same color. (We do not require that all three colors be used.) In how many ways can this be done?

6. In a classroom, 34 students are seated in 5 rows of 7 chairs. The place at the center of the room is unoccupied. A teacher decides to reassign the seats such that each student will occupy a chair adjacent to his/her present one (i.e. move one desk forward, back, left or right). In how many ways can this reassignment be made?

7. You have infinitely many boxes, and you randomly put 3 balls into them. The boxes are labeled $1, 2, \ldots$. Each ball has probability $1/2^n$ of being put into box n. The balls are placed independently of each other. What is the probability that some box will contain at least 2 balls?

8. For any subset $S \subseteq \{1, 2, \ldots, 15\}$, a number n is called an "anchor" for S if n and $n + |S|$ are both members of S, where $|S|$ denotes the number of members of S. Find the average number of anchors over all possible subsets $S \subseteq \{1, 2, \ldots, 15\}$.

9. At a certain college, there are 10 clubs and some number of students. For any two different students, there is some club such that exactly one of the two belongs to that club. For any three different students, there is some club such that either exactly one or all three belong to that club. What is the largest possible number of students?

10. A calculator has a display, which shows a nonnegative integer N, and a button, which replaces N by a random integer chosen uniformly from the set $\{0, 1, \ldots, N-1\}$, provided that $N > 0$. Initially, the display holds the number $N = 2003$. If the button is pressed repeatedly until $N = 0$, what is the probability that the numbers $1, 10, 100$, and 1000 will each show up on the display at some point?

Harvard-MIT Mathematics Tournament

March 15, 2003

Individual Round: Calculus Subject Test

1. A point is chosen randomly with uniform distribution in the interior of a circle of radius 1. What is its expected distance from the center of the circle?

2. A particle moves along the x-axis in such a way that its velocity at position x is given by the formula $v(x) = 2 + \sin x$. What is its acceleration at $x = \frac{\pi}{6}$?

3. What is the area of the region bounded by the curves $y = x^{2003}$ and $y = x^{1/2003}$ and lying above the x-axis?

4. The sequence of real numbers $x_1, x_2, x_3, \ldots$ satisfies $\lim_{n\to\infty}(x_{2n} + x_{2n+1}) = 315$ and $\lim_{n\to\infty}(x_{2n} + x_{2n-1}) = 2003$. Evaluate $\lim_{n\to\infty}(x_{2n}/x_{2n+1})$.

5. Find the minimum distance from the point $(0, 5/2)$ to the graph of $y = x^4/8$.

6. For n an integer, evaluate
$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2-0^2}} + \frac{1}{\sqrt{n^2-1^2}} + \cdots + \frac{1}{\sqrt{n^2-(n-1)^2}}\right).$$

7. For what value of $a > 1$ is
$$\int_a^{a^2} \frac{1}{x} \log \frac{x-1}{32}\, dx$$
minimum?

8. A right circular cone with a height of 12 inches and a base radius of 3 inches is filled with water and held with its vertex pointing downward. Water flows out through a hole at the vertex at a rate in cubic inches per second numerically equal to the height of the water in the cone. (For example, when the height of the water in the cone is 4 inches, water flows out at a rate of 4 cubic inches per second.) Determine how many seconds it will take for all of the water to flow out of the cone.

9. Two differentiable real functions $f(x)$ and $g(x)$ satisfy
$$\frac{f'(x)}{g'(x)} = e^{f(x)-g(x)}$$
for all x, and $f(0) = g(2003) = 1$. Find the largest constant c such that $f(2003) > c$ for all such functions f, g.

10. Evaluate
$$\int_{-\infty}^{\infty} \frac{1-x^2}{1+x^4}\, dx.$$

Harvard-MIT Mathematics Tournament

March 15, 2003

Individual Round: General Test, Part 1

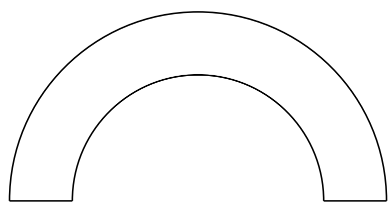
1. 10 people are playing musical chairs with n chairs in a circle. They can be seated in 7! ways (assuming only one person fits on each chair, of course), where different arrangements of the same people on chairs, even rotations, are considered different. Find n.

2. $OPEN$ is a square, and T is a point on side NO, such that triangle TOP has area 62 and triangle TEN has area 10. What is the length of a side of the square?

3. There are 16 members on the Height-Measurement Matching Team. Each member was asked, "How many other people on the team — not counting yourself — are exactly the same height as you?" The answers included six 1's, six 2's, and three 3's. What was the sixteenth answer? (Assume that everyone answered truthfully.)

4. How many 2-digit positive integers have an even number of positive divisors?

5. A room is built in the shape of the region between two semicircles with the same center and parallel diameters. The farthest distance between two points with a clear line of sight is 12m. What is the area (in m^2) of the room?



6. In how many ways can 3 bottles of ketchup and 7 bottles of mustard be arranged in a row so that no bottle of ketchup is immediately between two bottles of mustard? (The bottles of ketchup are mutually indistinguishable, as are the bottles of mustard.)

7. Find the real value of x such that $x^3 + 3x^2 + 3x + 7 = 0$.

8. A broken calculator has the $+$ and $\times$ keys switched. For how many ordered pairs (a, b) of integers will it correctly calculate $a + b$ using the labelled $+$ key?

9. Consider a 2003-gon inscribed in a circle and a triangulation of it with diagonals intersecting only at vertices. What is the smallest possible number of obtuse triangles in the triangulation?

10. Bessie the cow is trying to navigate her way through a field. She can travel only from lattice point to adjacent lattice point, can turn only at lattice points, and can travel only to the east or north. (A lattice point is a point whose coordinates are both integers.) $(0, 0)$ is the southwest corner of the field. $(5, 5)$ is the northeast corner of the field. Due to large rocks, Bessie is unable to walk on the points $(1, 1)$, $(2, 3)$, or $(3, 2)$. How many ways are there for Bessie to travel from $(0, 0)$ to $(5, 5)$ under these constraints?

Harvard-MIT Mathematics Tournament

March 15, 2003

Individual Round: General Test, Part 2

1. A compact disc has the shape of a circle of diameter 5 inches with a 1-inch-diameter circular hole in the center. Assuming the capacity of the CD is proportional to its area, how many inches would need to be added to the outer diameter to double the capacity?

2. You have a list of real numbers, whose sum is 40. If you replace every number x on the list by $1-x$, the sum of the new numbers will be 20. If instead you had replaced every number x by $1+x$, what would the sum then be?

3. How many positive rational numbers less than π have denominator at most 7 when written in lowest terms? (Integers have denominator 1.)

4. In triangle ABC with area 51, points D and E trisect AB and points F and G trisect BC. Find the largest possible area of quadrilateral $DEFG$.

5. You are given a 10×2 grid of unit squares. Two different squares are adjacent if they share a side. How many ways can one mark exactly nine of the squares so that no two marked squares are adjacent?

6. The numbers 112, 121, 123, 153, 243, 313, and 322 are among the rows, columns, and diagonals of a 3×3 square grid of digits (rows and diagonals read left-to-right, and columns read top-to-bottom). What 3-digit number completes the list?

7. Daniel and Scott are playing a game where a player wins as soon as he has two points more than his opponent. Both players start at par, and points are earned one at a time. If Daniel has a 60% chance of winning each point, what is the probability that he will win the game?

8. If $x \geq 0$, $y \geq 0$ are integers, randomly chosen with the constraint $x + y \leq 10$, what is the probability that $x + y$ is even?

9. In a classroom, 34 students are seated in 5 rows of 7 chairs. The place at the center of the room is unoccupied. A teacher decides to reassign the seats such that each student will occupy a chair adjacent to his/her present one (i.e. move one desk forward, back, left or right). In how many ways can this reassignment be made?

10. Several positive integers are given, not necessarily all different. Their sum is 2003. Suppose that n_1 of the given numbers are equal to 1, n_2 of them are equal to 2, …, n_{2003} of them are equal to 2003. Find the largest possible value of

$$n_2 + 2n_3 + 3n_4 + \cdots + 2002n_{2003}.$$

Harvard-MIT Mathematics Tournament

March 15, 2003

Guts Round

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HARVARD-MIT MATHEMATICS TOURNAMENT, MARCH 15, 2003 — GUTS ROUND

1. [**5**] Simplify $\sqrt[2003]{2\sqrt{11}-3\sqrt{5}} \cdot \sqrt[4006]{89+12\sqrt{55}}$.

2. [**5**] The graph of $x^4 = x^2y^2$ is a union of n different lines. What is the value of n?

3. [**5**] If a and b are positive integers that can each be written as a sum of two squares, then ab is also a sum of two squares. Find the smallest positive integer c such that $c = ab$, where $a = x^3 + y^3$ and $b = x^3 + y^3$ each have solutions in integers (x, y), but $c = x^3 + y^3$ does not.

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HARVARD-MIT MATHEMATICS TOURNAMENT, MARCH 15, 2003 — GUTS ROUND

4. [**6**] Let $z = 1 - 2i$. Find $\frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots$.

5. [**6**] Compute the surface area of a cube inscribed in a sphere of surface area π.

6. [**6**] Define the Fibonacci numbers by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. For how many n, $0 \leq n \leq 100$, is F_n a multiple of 13?

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HARVARD-MIT MATHEMATICS TOURNAMENT, MARCH 15, 2003 — GUTS ROUND

7. [**6**] a and b are integers such that $a + \sqrt{b} = \sqrt{15 + \sqrt{216}}$. Compute a/b.

8. [**6**] How many solutions in nonnegative integers (a, b, c) are there to the equation

$$2^a + 2^b = c! \quad ?$$

9. [**6**] For x a real number, let $f(x) = 0$ if $x < 1$ and $f(x) = 2x - 2$ if $x \geq 1$. How many solutions are there to the equation

$$f(f(f(f(x)))) = x?$$

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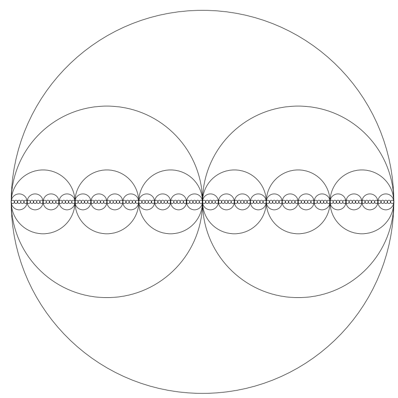
HARVARD-MIT MATHEMATICS TOURNAMENT, MARCH 15, 2003 — GUTS ROUND

10. [**7**] Suppose that A, B, C, D are four points in the plane, and let Q, R, S, T, U, V be the respective midpoints of AB, AC, AD, BC, BD, CD. If $QR = 2001, SU = 2002, TV = 2003$, find the distance between the midpoints of QU and RV.

11. [**7**] Find the smallest positive integer n such that $1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2$ is divisible by 100.

12. [**7**] As shown in the figure, a circle of radius 1 has two equal circles whose diameters cover a chosen diameter of the larger circle. In each of these smaller circles we similarly draw three equal circles, then four in each of those, and so on. Compute the area of the region enclosed by a positive even number of circles.



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HARVARD-MIT MATHEMATICS TOURNAMENT, MARCH 15, 2003 — GUTS ROUND

13. [**7**] If $xy = 5$ and $x^2 + y^2 = 21$, compute $x^4 + y^4$.

14. [**7**] A positive integer will be called "sparkly" if its smallest (positive) divisor, other than 1, equals the total number of divisors (including 1). How many of the numbers $2, 3, \ldots, 2003$ are sparkly?

15. [**7**] The product of the digits of a 5-digit number is 180. How many such numbers exist?

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HARVARD-MIT MATHEMATICS TOURNAMENT, MARCH 15, 2003 — GUTS ROUND

16. [**8**] What fraction of the area of a regular hexagon of side length 1 is within distance $\frac{1}{2}$ of at least one of the vertices?

17. [**8**] There are 10 cities in a state, and some pairs of cities are connected by roads. There are 40 roads altogether. A city is called a "hub" if it is directly connected to every other city. What is the largest possible number of hubs?

18. [**8**] Find the sum of the reciprocals of all the (positive) divisors of 144.

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HARVARD-MIT MATHEMATICS TOURNAMENT, MARCH 15, 2003 — GUTS ROUND

19. [**8**] Let r, s, t be the solutions to the equation $x^3 + ax^2 + bx + c = 0$. What is the value of $(rs)^2 + (st)^2 + (rt)^2$ in terms of a, b, and c?

20. [**8**] What is the smallest number of regular hexagons of side length 1 needed to completely cover a disc of radius 1?

21. [**8**] r and s are integers such that

$$3r \geq 2s - 3 \text{ and } 4s \geq r + 12.$$

What is the smallest possible value of r/s?

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HARVARD-MIT MATHEMATICS TOURNAMENT, MARCH 15, 2003 — GUTS ROUND

22. [**9**] There are 100 houses in a row on a street. A painter comes and paints every house red. Then, another painter comes and paints every third house (starting with house number 3) blue. Another painter comes and paints every fifth house red (even if it is already red), then another painter paints every seventh house blue, and so forth, alternating between red and blue, until 50 painters have been by. After this is finished, how many houses will be red?

23. [**9**] How many lattice points are enclosed by the triangle with vertices $(0, 99)$, $(5, 100)$, and $(2003, 500)$? Don't count boundary points.

24. [**9**] Compute the radius of the inscribed circle of a triangle with sides 15, 16, and 17.

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25. [**9**] Let ABC be an isosceles triangle with apex A. Let I be the incenter. If $AI = 3$ and the distance from I to BC is 2, then what is the length of BC?

26. [**9**] Find all integers x such that $x^2 + 6x + 28$ is a perfect square.

27. [**9**] The rational numbers x and y, when written in lowest terms, have denominators 60 and 70, respectively. What is the smallest possible denominator of $x + y$?

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28. [**10**] A point in three-space has distances $2, 6, 7, 8, 9$ from five of the vertices of a regular octahedron. What is its distance from the sixth vertex?

29. [**10**] A *palindrome* is a positive integer that reads the same backwards as forwards, such as 82328. What is the smallest 5-digit palindrome that is a multiple of 99?

30. [**10**] The sequence $a_1, a_2, a_3, \ldots$ of real numbers satisfies the recurrence

$$a_{n+1} = \frac{a_n^2 - a_{n-1} + 2a_n}{a_{n-1} + 1}.$$

Given that $a_1 = 1$ and $a_9 = 7$, find a_5.

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31. [**10**] A cylinder of base radius 1 is cut into two equal parts along a plane passing through the center of the cylinder and tangent to the two base circles. Suppose that each piece's surface area is m times its volume. Find the greatest lower bound for all possible values of m as the height of the cylinder varies.

32. [**10**] If x, y, and z are real numbers such that $2x^2 + y^2 + z^2 = 2x - 4y + 2xz - 5$, find the maximum possible value of $x - y + z$.

33. [**10**] We are given triangle ABC, with $AB = 9$, $AC = 10$, and $BC = 12$, and a point D on BC. B and C are reflected in AD to B' and C', respectively. Suppose that lines BC' and $B'C$ never meet (i.e., are parallel and distinct). Find BD.

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34. **[12]** $OKRA$ is a trapezoid with OK parallel to RA. If $OK = 12$ and RA is a positive integer, how many integer values can be taken on by the length of the segment in the trapezoid, parallel to OK, through the intersection of the diagonals?

35. **[12]** A certain lottery has tickets labeled with the numbers $1, 2, 3, \ldots, 1000$. The lottery is run as follows: First, a ticket is drawn at random. If the number on the ticket is odd, the drawing ends; if it is even, another ticket is randomly drawn (without replacement). If this new ticket has an odd number, the drawing ends; if it is even, another ticket is randomly drawn (again without replacement), and so forth, until an odd number is drawn. Then, every person whose ticket number was drawn (at any point in the process) wins a prize.

    You have ticket number 1000. What is the probability that you get a prize?

36. **[12]** A teacher must divide 221 apples evenly among 403 students. What is the minimal number of pieces into which she must cut the apples? (A whole uncut apple counts as one piece.)

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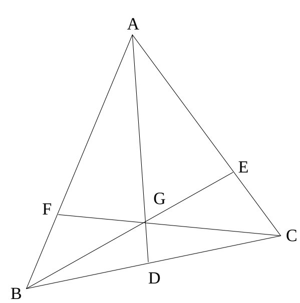
37. **[15]** A *quagga* is an extinct chess piece whose move is like a knight's, but much longer: it can move 6 squares in any direction (up, down, left, or right) and then 5 squares in a perpendicular direction. Find the number of ways to place 51 quaggas on an 8×8 chessboard in such a way that no quagga attacks another. (Since quaggas are naturally belligerent creatures, a quagga is considered to attack quaggas on any squares it can move to, as well as any other quaggas on the same square.)

38. **[15]** Given are real numbers x, y. For any pair of real numbers a_0, a_1, define a sequence by $a_{n+2} = xa_{n+1} + ya_n$ for $n \geq 0$. Suppose that there exists a fixed nonnegative integer m such that, for every choice of a_0 and a_1, the numbers a_m, a_{m+1}, a_{m+3}, in this order, form an arithmetic progression. Find all possible values of y.

39. **[15]** In the figure, if $AE = 3$, $CE = 1$, $BD = CD = 2$, and $AB = 5$, find AG.



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40. **[18]** All the sequences consisting of five letters from the set $\{T, U, R, N, I, P\}$ (with repetitions allowed) are arranged in alphabetical order in a dictionary. Two sequences are called "anagrams" of each other if one can be obtained by rearranging the letters of the other. How many pairs of anagrams are there that have exactly 100 other sequences between them in the dictionary?

41. **[18]** A hotel consists of a 2×8 square grid of rooms, each occupied by one guest. All the guests are uncomfortable, so each guest would like to move to one of the adjoining rooms (horizontally or vertically). Of course, they should do this simultaneously, in such a way that each room will again have one guest. In how many different ways can they collectively move?

42. **[18]** A tightrope walker stands in the center of a rope of length 32 meters. Every minute she walks forward one meter with probability $3/4$ and backward one meter with probability $1/4$. What is the probability that she reaches the end in front of her before the end behind her?

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43. Write down an integer N between 0 and 10, inclusive. You will receive N points — unless some other team writes down the same N, in which case you receive nothing.

44. A *partition* of a number n is a sequence of positive integers, arranged in nonincreasing order, whose sum is n. For example, $n = 4$ has 5 partitions: $1+1+1+1 = 2+1+1 = 2+2 = 3+1 = 4$. Given two different partitions of the same number, $n = a_1 + a_2 + \cdots + a_k = b_1 + b_2 + \cdots + b_l$, where $k \leq l$, the first partition is said to *dominate* the second if all of the following inequalities hold:

$$\begin{aligned} a_1 &\geq b_1; \\ a_1 + a_2 &\geq b_1 + b_2; \\ a_1 + a_2 + a_3 &\geq b_1 + b_2 + b_3; \\ &\vdots \\ a_1 + a_2 + \cdots + a_k &\geq b_1 + b_2 + \cdots + b_k. \end{aligned}$$

Find as many partitions of the number $n = 20$ as possible such that none of the partitions dominates any other. Your score will be the number of partitions you find. If you make a mistake and one of your partitions does dominate another, your score is the largest m such that the first m partitions you list constitute a valid answer.

45. Find a set S of positive integers such that no two distinct subsets of S have the same sum. Your score will be $\lfloor 20(2^n/r - 2) \rfloor$, where n is the number of elements in the set S, and r is the largest element of S (assuming, of course, that this number is nonnegative).

Hej då!

Harvard-MIT Mathematics Tournament

March 15, 2003

Team Round

Completions and Configurations

Given a set A and a nonnegative integer k, the *k-completion* of A is the collection of all k-element subsets of A, and a *k-configuration* of A is any subset of the k-completion of A (including the empty set and the entire k-completion). For instance, the 2-completion of $A = \{1, 2, 3\}$ is $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, and the 2-configurations of A are

$$\begin{array}{cc} \{\} & \{\{1,2\}\} \\ \{\{1,3\}\} & \{\{2,3\}\} \\ \{\{1,2\},\{1,3\}\} & \{\{1,2\},\{2,3\}\} \\ \{\{1,3\},\{2,3\}\} & \{\{1,2\},\{1,3\},\{2,3\}\} \end{array}$$

The *order* of an element a of A with respect to a given k-configuration of A is the number of subsets in the k-configuration that contain a. A k-configuration of a set A is *consistent* if the order of every element of A is the same, and the *order* of a consistent k-configuration is this common value.

1. (a) **[10]** How many k-configurations are there of a set that has n elements?

   (b) **[10]** How many k-configurations that have m elements are there of a set that has n elements?

2. **[15]** Suppose A is a set with n elements, and k is a divisor of n. Find the number of consistent k-configurations of A of order 1.

3. (a) **[15]** Let $A_n = \{a_1, a_2, a_3, \ldots, a_n, b\}$, for $n \geq 3$, and let C_n be the 2-configuration consisting of $\{a_i, a_{i+1}\}$ for all $1 \leq i \leq n-1$, $\{a_1, a_n\}$, and $\{a_i, b\}$ for $1 \leq i \leq n$. Let $S_e(n)$ be the number of subsets of C_n that are consistent of order e. Find $S_e(101)$ for $e = 1, 2$, and 3.

   (b) **[20]** Let $A = \{V, W, X, Y, Z, v, w, x, y, z\}$. Find the number of subsets of the 2-configuration

   $$\{\ \{V,W\},\{W,X\},\{X,Y\},\{Y,Z\},\{Z,V\},\ \{v,x\},\{v,y\},\{w,y\},\{w,z\},\{x,z\},$$

   $$\{V,v\},\{W,w\},\{X,x\},\{Y,y\},\{Z,z\}\ \}$$

   that are consistent of order 1.

   (c) **[30]** Let $A = \{a_1, b_1, a_2, b_2, \ldots, a_{10}, b_{10}\}$, and consider the 2-configuration C consisting of $\{a_i, b_i\}$ for all $1 \leq i \leq 10$, $\{a_i, a_{i+1}\}$ for all $1 \leq i \leq 9$, and $\{b_i, b_{i+1}\}$ for all $1 \leq i \leq 9$. Find the number of subsets of C that are consistent of order 1.

Define a k-configuration of A to be *m-separable* if we can label each element of A with an integer from 1 to m (inclusive) so that there is no element E of the k-configuration all of whose elements are assigned the same integer. If C is any subset of A, then C is *m-separable* if we can assign an integer from 1 to m to each element of C so that there is no element E of the k-configuration such that $E \subseteq C$ and all elements of E are assigned the same integer.

4. (a) **[15]** Suppose A has n elements, where $n \geq 2$, and C is a 2-configuration of A that is not m-separable for any $m < n$. What is (in terms of n) the smallest number of elements that C can have?

   (b) **[15]** Show that every 3-configuration of an n-element set A is m-separable for every integer $m \geq n/2$.

   (c) **[25]** Fix $k \geq 2$, and suppose A has k^2 elements. Show that any k-configuration of A with fewer than $\binom{k^2-1}{k-1}$ elements is k-separable.

5. **[30]** Let $B_k(n)$ be the largest number of elements in a 2-separable k-configuration of a set with $2n$ elements ($2 \leq k \leq n$). Find a closed-form expression (i.e. an expression not involving any sums or products with an variable number of terms) for $B_k(n)$.

6. **[40]** Prove that any 2-configuration containing e elements is m-separable for some $m \leq \frac{1}{2} + \sqrt{2e + \frac{1}{4}}$.

A *cell* of a 2-configuration of a set A is a nonempty subset C of A such that

i. for any two distinct elements a, b of C, there exists a sequence $c_0, c_1, \ldots, c_n$ of elements of A with $c_0 = a, c_n = b$, and such that $\{c_0, c_1\}, \{c_1, c_2\}, \ldots, \{c_{n-1}, c_n\}$ are all elements of the 2-configuration, and

ii. if a is an element of C and b is an element of A but not of C, there does NOT exist a sequence $c_0, c_1, \ldots, c_n$ of elements of A with $c_0 = a, c_n = b$, and such that $\{c_0, c_1\}, \{c_1, c_2\}, \ldots, \{c_{n-1}, c_n\}$ are all elements of the 2-configuration.

Also, we define a 2-configuration of A to be *barren* if there is no subset $\{a_0, a_1, \ldots, a_n\}$ of A, with $n \geq 2$, such that $\{a_0, a_1\}, \{a_1, a_2\}, \ldots, \{a_{n-1}, a_n\}$ and $\{a_n, a_0\}$ are all elements of the 2-configuration.

7. **[20]** Show that, given any 2-configuration of a set A, every element of A belongs to exactly one cell.

8. (a) **[15]** Given a set A with $n \geq 1$ elements, find the number of consistent 2-configurations of A of order 1 with exactly 1 cell.

   (b) **[25]** Given a set A with 10 elements, find the number of consistent 2-configurations of A of order 2 with exactly 1 cell.

   (c) **[25]** Given a set A with 10 elements, find the number of consistent 2-configurations of order 2 with exactly 2 cells.

9. (a) **[15]** Show that if every cell of a 2-configuration of a finite set A is m-separable, then the whole 2-configuration is m-separable.

   (b) **[30]** Show that any barren 2-configuration of a finite set A is 2-separable.

10. **[45]** Show that every consistent 2-configuration of order 4 on a finite set A has a subset that is a consistent 2-configuration of order 2.

Harvard-MIT Mathematics Tournament

February 28, 2004

Individual Round: Algebra Subject Test

1. How many ordered pairs of integers (a,b) satisfy all of the following inequalities?

$$\begin{aligned} a^2+b^2 &< 16 \\ a^2+b^2 &< 8a \\ a^2+b^2 &< 8b \end{aligned}$$

2. Find the largest number n such that (2004!)! is divisible by $((n!)!)!$.

3. Compute:

$$\left\lfloor \frac{2005^3}{2003 \cdot 2004} - \frac{2003^3}{2004 \cdot 2005} \right\rfloor.$$

4. Evaluate the sum

$$\frac{1}{2\lfloor\sqrt{1}\rfloor+1} + \frac{1}{2\lfloor\sqrt{2}\rfloor+1} + \frac{1}{2\lfloor\sqrt{3}\rfloor+1} + \cdots + \frac{1}{2\lfloor\sqrt{100}\rfloor+1}.$$

5. There exists a positive real number x such that $\cos(\tan^{-1}(x)) = x$. Find the value of x^2.

6. Find all real solutions to $x^4 + (2-x)^4 = 34$.

7. If x, y, k are positive reals such that

$$3 = k^2\left(\frac{x^2}{y^2} + \frac{y^2}{x^2}\right) + k\left(\frac{x}{y} + \frac{y}{x}\right),$$

find the maximum possible value of k.

8. Let x be a real number such that $x^3 + 4x = 8$. Determine the value of $x^7 + 64x^2$.

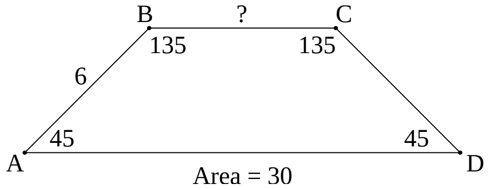
9. A sequence of positive integers is defined by $a_0 = 1$ and $a_{n+1} = a_n^2 + 1$ for each $n \geq 0$. Find $\gcd(a_{999}, a_{2004})$.

10. There exists a polynomial P of degree 5 with the following property: if z is a complex number such that $z^5 + 2004z = 1$, then $P(z^2) = 0$. Calculate the quotient $P(1)/P(-1)$.

Harvard-MIT Mathematics Tournament
February 28, 2004

Individual Round: Geometry Subject Test

1. In trapezoid $ABCD$, AD is parallel to BC. $\angle A = \angle D = 45°$, while $\angle B = \angle C = 135°$. If $AB = 6$ and the area of $ABCD$ is 30, find BC.



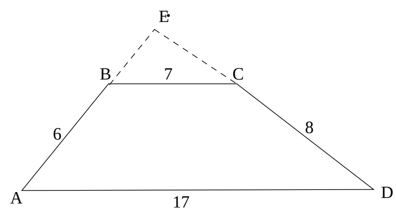
2. A parallelogram has 3 of its vertices at (1, 2), (3,8), and (4, 1). Compute the sum of the possible x-coordinates for the 4th vertex.

3. A swimming pool is in the shape of a circle with diameter 60 ft. The depth varies linearly along the east-west direction from 3 ft at the shallow end in the east to 15 ft at the diving end in the west (this is so that divers look impressive against the sunset) but does not vary at all along the north-south direction. What is the volume of the pool, in ft^3?

4. P is inside rectangle $ABCD$. $PA = 2, PB = 3$, and $PC = 10$. Find PD.

5. Find the area of the region of the xy-plane defined by the inequality $|x|+|y|+|x+y| \le 1$.

6. In trapezoid $ABCD$ shown, AD is parallel to BC, and $AB = 6, BC = 7, CD = 8, AD = 17$. If sides AB and CD are extended to meet at E, find the resulting angle at E (in degrees).



7. Yet another trapezoid $ABCD$ has AD parallel to BC. AC and BD intersect at P. If $[ADP]/[BCP] = 1/2$, find $[ADP]/[ABCD]$. (Here the notation $[P_1 \cdots P_n]$ denotes the area of the polygon $P_1 \cdots P_n$.)

8. A triangle has side lengths 18, 24, and 30. Find the area of the triangle whose vertices are the incenter, circumcenter, and centroid of the original triangle.

9. Given is a regular tetrahedron of volume 1. We obtain a second regular tetrahedron by reflecting the given one through its center. What is the volume of their intersection?

10. Right triangle XYZ has right angle at Y and $XY = 228$, $YZ = 2004$. Angle Y is trisected, and the angle trisectors intersect XZ at P and Q so that X, P, Q, Z lie on XZ in that order. Find the value of $(PY + YZ)(QY + XY)$.

Harvard-MIT Mathematics Tournament

February 28, 2004

Individual Round: Combinatorics Subject Test

1. There are 1000 rooms in a row along a long corridor. Initially the first room contains 1000 people and the remaining rooms are empty. Each minute, the following happens: for each room containing more than one person, someone in that room decides it is too crowded and moves to the next room. All these movements are simultaneous (so nobody moves more than once within a minute). After one hour, how many different rooms will have people in them?

2. How many ways can you mark 8 squares of an 8×8 chessboard so that no two marked squares are in the same row or column, and none of the four corner squares is marked? (Rotations and reflections are considered different.)

3. A class of 10 students took a math test. Each problem was solved by exactly 7 of the students. If the first nine students each solved 4 problems, how many problems did the tenth student solve?

4. Andrea flips a fair coin repeatedly, continuing until she either flips two heads in a row (the sequence HH) or flips tails followed by heads (the sequence TH). What is the probability that she will stop after flipping HH?

5. A best-of-9 series is to be played between two teams; that is, the first team to win 5 games is the winner. The Mathletes have a chance of $2/3$ of winning any given game. What is the probability that exactly 7 games will need to be played to determine a winner?

6. A committee of 5 is to be chosen from a group of 9 people. How many ways can it be chosen, if Bill and Karl must serve together or not at all, and Alice and Jane refuse to serve with each other?

7. We have a polyhedron such that an ant can walk from one vertex to another, traveling only along edges, and traversing every edge exactly once. What is the smallest possible total number of vertices, edges, and faces of this polyhedron?

8. Urn A contains 4 white balls and 2 red balls. Urn B contains 3 red balls and 3 black balls. An urn is randomly selected, and then a ball inside of that urn is removed. We then repeat the process of selecting an urn and drawing out a ball, without returning the first ball. What is the probability that the first ball drawn was red, given that the second ball drawn was black?

9. A classroom consists of a 5×5 array of desks, to be filled by anywhere from 0 to 25 students, inclusive. No student will sit at a desk unless either all other desks in its row or all others in its column are filled (or both). Considering only the set of desks that are occupied (and not which student sits at each desk), how many possible arrangements are there?

10. In a game similar to three card monte, the dealer places three cards on the table: the queen of spades and two red cards. The cards are placed in a row, and the queen starts in the center; the card configuration is thus RQR. The dealer proceeds to move. With each move, the dealer randomly switches the center card with one of the two edge cards (so the configuration after the first move is either RRQ or QRR). What is the probability that, after 2004 moves, the center card is the queen?

Harvard-MIT Mathematics Tournament
February 28, 2004

Individual Round: Calculus Subject Test

1. Let $f(x) = \sin(\sin x)$. Evaluate $\lim_{h\to 0} \frac{f(x+h)-f(h)}{x}$ at $x = \pi$.

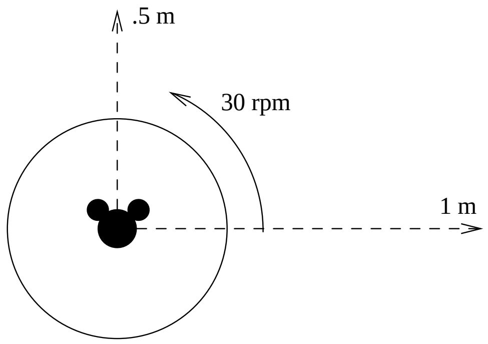
2. Suppose the function $f(x) - f(2x)$ has derivative 5 at $x = 1$ and derivative 7 at $x = 2$. Find the derivative of $f(x) - f(4x)$ at $x = 1$.

3. Find $\lim_{x\to\infty}(\sqrt[3]{x^3 + x^2} - \sqrt[3]{x^3 - x^2})$.

4. Let $f(x) = \cos(\cos(\cos(\cos(\cos(\cos(\cos(\cos x)))))))$, and suppose that the number a satisfies the equation $a = \cos a$. Express $f'(a)$ as a polynomial in a.

5. A mouse is sitting in a toy car on a negligibly small turntable. The car cannot turn on its own, but the mouse can control when the car is launched and when the car stops (the car has brakes). When the mouse chooses to launch, the car will immediately leave the turntable on a straight trajectory at 1 meter per second.

   Suddenly someone turns on the turntable; it spins at 30 rpm. Consider the set S of points the mouse can reach in his car within 1 second after the turntable is set in motion. (For example, the arrows in the figure below represent two possible paths the mouse can take.) What is the area of S, in square meters?



6. For $x > 0$, let $f(x) = x^x$. Find all values of x for which $f(x) = f'(x)$.

7. Find the area of the region in the xy-plane satisfying $x^6 - x^2 + y^2 \leq 0$.

8. If x and y are real numbers with $(x+y)^4 = x - y$, what is the maximum possible value of y?

9. Find the positive constant c_0 such that the series

$$\sum_{n=0}^{\infty} \frac{n!}{(cn)^n}$$

   converges for $c > c_0$ and diverges for $0 < c < c_0$.

10. Let $P(x) = x^3 - \frac{3}{2}x^2 + x + \frac{1}{4}$. Let $P^{[1]}(x) = P(x)$, and for $n \geq 1$, let $P^{[n+1]}(x) = P^{[n]}(P(x))$. Evaluate $\int_0^1 P^{[2004]}(x)\,dx$.

Harvard-MIT Mathematics Tournament

February 28, 2004

Individual Round: General Test, Part 1

1. There are 1000 rooms in a row along a long corridor. Initially the first room contains 1000 people and the remaining rooms are empty. Each minute, the following happens: for each room containing more than one person, someone in that room decides it is too crowded and moves to the next room. All these movements are simultaneous (so nobody moves more than once within a minute). After one hour, how many different rooms will have people in them?

2. What is the largest whole number that is equal to the product of its digits?

3. Suppose f is a function that assigns to each real number x a value $f(x)$, and suppose the equation

$$f(x_1 + x_2 + x_3 + x_4 + x_5) = f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) - 8$$

holds for all real numbers x_1, x_2, x_3, x_4, x_5. What is $f(0)$?

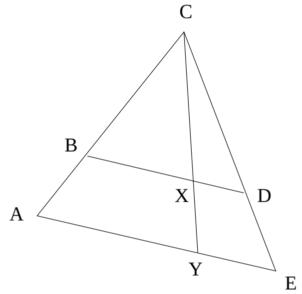
4. How many ways can you mark 8 squares of an 8×8 chessboard so that no two marked squares are in the same row or column, and none of the four corner squares is marked? (Rotations and reflections are considered different.)

5. A rectangle has perimeter 10 and diagonal $\sqrt{15}$. What is its area?

6. Find the ordered quadruple of digits (A, B, C, D), with $A > B > C > D$, such that

$$\begin{array}{rr} & ABCD \\ - & \underline{DCBA} \\ = & BDAC. \end{array}$$

7. Let ACE be a triangle with a point B on segment AC and a point D on segment CE such that BD is parallel to AE. A point Y is chosen on segment AE, and segment CY is drawn. Let X be the intersection of CY and BD. If $CX = 5$, $XY = 3$, what is the ratio of the area of trapezoid $ABDE$ to the area of triangle BCD?

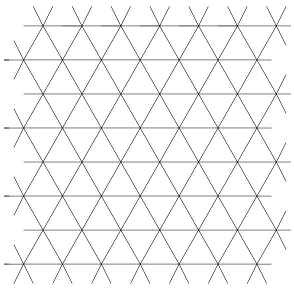


8. You have a 10×10 grid of squares. You write a number in each square as follows: you write $1, 2, 3, \ldots, 10$ from left to right across the top row, then $11, 12, \ldots, 20$ across the second row, and so on, ending with a 100 in the bottom right square. You then write a second number in each square, writing $1, 2, \ldots, 10$ in the first column (from top to bottom), then $11, 12, \ldots, 20$ in the second column, and so forth.

   When this process is finished, how many squares will have the property that their two numbers sum to 101?

9. Urn A contains 4 white balls and 2 red balls. Urn B contains 3 red balls and 3 black balls. An urn is randomly selected, and then a ball inside of that urn is removed. We then repeat the process of selecting an urn and drawing out a ball, without returning the first ball. What is the probability that the first ball drawn was red, given that the second ball drawn was black?

10. A floor is tiled with equilateral triangles of side length 1, as shown. If you drop a needle of length 2 somewhere on the floor, what is the largest number of triangles it could end up intersecting? (Only count the triangles whose interiors are met by the needle — touching along edges or at corners doesn't qualify.)



Harvard-MIT Mathematics Tournament

February 28, 2004

Individual Round: General Test, Part 2

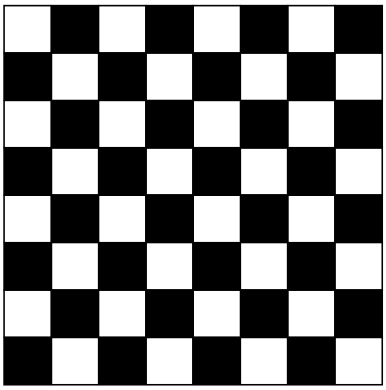
1. Find the largest number n such that $(2004!)!$ is divisible by $((n!)!)!$.

2. Andrea flips a fair coin repeatedly, continuing until she either flips two heads in a row (the sequence HH) or flips tails followed by heads (the sequence TH). What is the probability that she will stop after flipping HH?

3. How many ordered pairs of integers (a, b) satisfy all of the following inequalities?

$$\begin{aligned} a^2 + b^2 &< 16 \\ a^2 + b^2 &< 8a \\ a^2 + b^2 &< 8b \end{aligned}$$

4. A horse stands at the corner of a chessboard, a white square. With each jump, the horse can move either two squares horizontally and one vertically or two vertically and one horizontally (like a knight moves). The horse earns two carrots every time it lands on a black square, but it must pay a carrot in rent to rabbit who owns the chessboard for every move it makes. When the horse reaches the square on which it began, it can leave. What is the maximum number of carrots the horse can earn without touching any square more than twice?



5. Eight strangers are preparing to play bridge. How many ways can they be grouped into two bridge games — that is, into unordered pairs of unordered pairs of people?

6. a and b are positive integers. When written in binary, a has 2004 1's, and b has 2005 1's (not necessarily consecutive). What is the smallest number of 1's $a + b$ could possibly have?

7. Farmer John is grazing his cows at the origin. There is a river that runs east to west 50 feet north of the origin. The barn is 100 feet to the south and 80 feet to the east of the origin. Farmer John leads his cows to the river to take a swim, then the cows leave the river from the same place they entered and Farmer John leads them to the barn. He does this using the shortest path possible, and the total distance he travels is d feet. Find the value of d.

8. A freight train leaves the town of Jenkinsville at 1:00 PM traveling due east at constant speed. Jim, a hobo, sneaks onto the train and falls asleep. At the same time, Julie leaves Jenkinsville on her bicycle, traveling along a straight road in a northeasterly direction (but not due northeast) at 10 miles per hour. At 1:12 PM, Jim rolls over in his sleep and falls from the train onto the side of the tracks. He wakes up and immediately begins walking at 3.5 miles per hour directly towards the road on which Julie is riding. Jim reaches the road at 2:12 PM, just as Julie is riding by. What is the speed of the train in miles per hour?

9. Given is a regular tetrahedron of volume 1. We obtain a second regular tetrahedron by reflecting the given one through its center. What is the volume of their intersection?

10. A *lattice point* is a point whose coordinates are both integers. Suppose Johann walks in a line from the point $(0, 2004)$ to a random lattice point in the interior (not on the boundary) of the square with vertices $(0,0),(0,99),(99,99),(99,0)$. What is the probability that his path, including the endpoints, contains an even number of lattice points?

Harvard-MIT Mathematics Tournament

February 28, 2004

Guts Round

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1. **[5]** Find the value of

$$\binom{6}{1}2^1 + \binom{6}{2}2^2 + \binom{6}{3}2^3 + \binom{6}{4}2^4 + \binom{6}{5}2^5 + \binom{6}{6}2^6.$$

2. **[5]** If the three points

$$(1, a, b)$$
$$(a, 2, b)$$
$$(a, b, 3)$$

are collinear (in 3-space), what is the value of $a + b$?

3. **[5]** If the system of equations

$$\begin{aligned} |x + y| &= 99 \\ |x - y| &= c \end{aligned}$$

has exactly two real solutions (x, y), find the value of c.

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4. **[6]** A tree grows in a rather peculiar manner. Lateral cross-sections of the trunk, leaves, branches, twigs, and so forth are circles. The trunk is 1 meter in diameter to a height of 1 meter, at which point it splits into two sections, each with diameter .5 meter. These sections are each one meter long, at which point they each split into two sections, each with diameter .25 meter. This continues indefinitely: every section of tree is 1 meter long and splits into two smaller sections, each with half the diameter of the previous.

   What is the total volume of the tree?

5. **[6]** Augustin has six $1 \times 2 \times \pi$ bricks. He stacks them, one on top of another, to form a tower six bricks high. Each brick can be in any orientation so long as it rests flat on top of the next brick below it (or on the floor). How many distinct heights of towers can he make?

6. **[6]** Find the smallest integer n such that $\sqrt{n + 99} - \sqrt{n} < 1$.

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7. **[6]** Find the shortest distance from the line $3x+4y=25$ to the circle $x^2+y^2=6x-8y$.

8. **[6]** I have chosen five of the numbers $\{1,2,3,4,5,6,7\}$. If I told you what their product was, that would not be enough information for you to figure out whether their sum was even or odd. What is their product?

9. **[6]** A positive integer n is *picante* if $n!$ ends in the same number of zeroes whether written in base 7 or in base 8. How many of the numbers $1,2,\ldots,2004$ are picante?

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10. **[7]** Let $f(x)=x^2+x^4+x^6+x^8+\cdots$, for all real x such that the sum converges. For how many real numbers x does $f(x)=x$?

11. **[7]** Find all numbers n with the following property: there is exactly one set of 8 different positive integers whose sum is n.

12. **[7]** A convex quadrilateral is drawn in the coordinate plane such that each of its vertices (x,y) satisfies the equations $x^2+y^2=73$ and $xy=24$. What is the area of this quadrilateral?

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13. **[7]** Find all positive integer solutions (m,n) to the following equation:

$$m^2 = 1! + 2! + \cdots + n!.$$

14. **[7]** If $a_1=1, a_2=0$, and $a_{n+1}=a_n+\frac{a_{n+2}}{2}$ for all $n\geq 1$, compute a_{2004}.

15. **[7]** A regular decagon $A_0A_1A_2\cdots A_9$ is given in the plane. Compute $\angle A_0A_3A_7$ in degrees.

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16. [**8**] An n-string is a string of digits formed by writing the numbers $1, 2, \ldots, n$ in some order (in base ten). For example, one possible 10-string is

$$35728910461$$

What is the smallest $n > 1$ such that there exists a palindromic n-string?

17. [**8**] Kate has four red socks and four blue socks. If she randomly divides these eight socks into four pairs, what is the probability that none of the pairs will be mismatched? That is, what is the probability that each pair will consist either of two red socks or of two blue socks?

18. [**8**] On a spherical planet with diameter $10,000$ km, powerful explosives are placed at the north and south poles. The explosives are designed to vaporize all matter within $5,000$ km of ground zero and leave anything beyond $5,000$ km untouched. After the explosives are set off, what is the new surface area of the planet, in square kilometers?

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19. [**8**] The Fibonacci numbers are defined by $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. If the number

$$\frac{F_{2003}}{F_{2002}} - \frac{F_{2004}}{F_{2003}}$$

is written as a fraction in lowest terms, what is the numerator?

20. [**8**] Two positive rational numbers x and y, when written in lowest terms, have the property that the sum of their numerators is 9 and the sum of their denominators is 10. What is the largest possible value of $x + y$?

21. [**8**] Find all ordered pairs of integers (x, y) such that $3^x 4^y = 2^{x+y} + 2^{2(x+y)-1}$.

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22. [**9**] I have written a strictly increasing sequence of six positive integers, such that each number (besides the first) is a multiple of the one before it, and the sum of all six numbers is 79. What is the largest number in my sequence?

23. [**9**] Find the largest integer n such that $3^{512} - 1$ is divisible by 2^n.

24. [**9**] We say a point is *contained* in a square if it is in its interior or on its boundary. Three unit squares are given in the plane such that there is a point contained in all three. Furthermore, three points A, B, C, are given, each contained in at least one of the squares. Find the maximum area of triangle ABC.

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25. [**9**] Suppose $x^3 - ax^2 + bx - 48$ is a polynomial with three positive roots p, q, and r such that $p < q < r$. What is the minimum possible value of $1/p + 2/q + 3/r$?

26. [**9**] How many of the integers $1, 2, \ldots, 2004$ can be represented as $(mn + 1)/(m + n)$ for positive integers m and n?

27. [**9**] A regular hexagon has one side along the diameter of a semicircle, and the two opposite vertices on the semicircle. Find the area of the hexagon if the diameter of the semicircle is 1.

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28. [**10**] Find the value of

$$\binom{2003}{1} + \binom{2003}{4} + \binom{2003}{7} + \cdots + \binom{2003}{2002}.$$

29. [**10**] A regular dodecahedron is projected orthogonally onto a plane, and its image is an n-sided polygon. What is the smallest possible value of n?

30. [**10**] We have an n-gon, and each of its vertices is labeled with a number from the set $\{1, \ldots, 10\}$. We know that for any pair of distinct numbers from this set there is at least one side of the polygon whose endpoints have these two numbers. Find the smallest possible value of n.

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31. [**10**] P is a point inside triangle ABC, and lines AP, BP, CP intersect the opposite sides BC, CA, AB in points D, E, F, respectively. It is given that $\angle APB = 90°$, and that $AC = BC$ and $AB = BD$. We also know that $BF = 1$, and that $BC = 999$. Find AF.

32. [**10**] Define the sequence $b_0, b_1, \ldots, b_{59}$ by

$$b_i = \begin{cases} 1 & \text{if i is a multiple of 3} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{a_i\}$ be a sequence of elements of $\{0, 1\}$ such that

$$b_n \equiv a_{n-1} + a_n + a_{n+1} \pmod 2$$

for $0 \leq n \leq 59$ ($a_0 = a_{60}$ and $a_{-1} = a_{59}$). Find all possible values of $4a_0 + 2a_1 + a_2$.

33. [**10**] A plane P slices through a cube of volume 1 with a cross-section in the shape of a regular hexagon. This cube also has an inscribed sphere, whose intersection with P is a circle. What is the area of the region inside the regular hexagon but outside the circle?

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34. [**12**] Find the number of 20-tuples of integers $x_1, \ldots, x_{10}, y_1, \ldots, y_{10}$ with the following properties:

- $1 \leq x_i \leq 10$ and $1 \leq y_i \leq 10$ for each i;
- $x_i \leq x_{i+1}$ for $i = 1, \ldots, 9$;
- if $x_i = x_{i+1}$, then $y_i \leq y_{i+1}$.

35. [**12**] There are eleven positive integers n such that there exists a convex polygon with n sides whose angles, in degrees, are unequal integers that are in arithmetic progression. Find the sum of these values of n.

36. [**12**] For a string of P's and Q's, the *value* is defined to be the product of the positions of the P's. For example, the string $PPQPQQ$ has value $1 \cdot 2 \cdot 4 = 8$.

Also, a string is called *antipalindromic* if writing it backwards, then turning all the P's into Q's and vice versa, produces the original string. For example, $PPQPQQ$ is antipalindromic.

There are 2^{1002} antipalindromic strings of length 2004. Find the sum of the reciprocals of their values.

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37. **[15]** Simplify $\prod_{k=1}^{2004} \sin(2\pi k/4009)$.

38. **[15]** Let $S = \{p_1 p_2 \cdots p_n \mid p_1, p_2, \ldots, p_n$ are distinct primes and $p_1, \ldots, p_n < 30\}$. Assume 1 is in S. Let a_1 be an element of S. We define, for all positive integers n:

$$a_{n+1} = a_n/(n+1) \quad \text{if } a_n \text{ is divisible by } n+1;$$

$$a_{n+1} = (n+2)a_n \quad \text{if } a_n \text{ is not divisible by } n+1.$$

How many distinct possible values of a_1 are there such that $a_j = a_1$ for infinitely many j's?

39. **[15]** You want to arrange the numbers $1, 2, 3, \ldots, 25$ in a sequence with the following property: if n is divisible by m, then the nth number is divisible by the mth number. How many such sequences are there?

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40. **[18]** You would like to provide airline service to the 10 cities in the nation of Schizophrenia, by instituting a certain number of two-way routes between cities. Unfortunately, the government is about to divide Schizophrenia into two warring countries of five cities each, and you don't know which cities will be in each new country. All airplane service between the two new countries will be discontinued. However, you want to make sure that you set up your routes so that, for any two cities in the same new country, it will be possible to get from one city to the other (without leaving the country).

    What is the minimum number of routes you must set up to be assured of doing this, no matter how the government divides up the country?

41. **[18]** A tetrahedron has all its faces triangles with sides $13, 14, 15$. What is its volume?

42. **[18]** S is a set of complex numbers such that if $u, v \in S$, then $uv \in S$ and $u^2 + v^2 \in S$. Suppose that the number N of elements of S with absolute value at most 1 is finite. What is the largest possible value of N?

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43. Write down an integer from 0 to 20 inclusive. This problem will be scored as follows: if N is the second-largest number from among the responses submitted, then each team that submits N gets N points, and everyone else gets zero. (If every team picks the same number then nobody gets any points.)

44. Shown on your answer sheet is a 20×20 grid. Place as many queens as you can so that each of them attacks at most one other queen. (A queen is a chess piece that can move any number of squares horizontally, vertically, or diagonally.) It's not very hard to get 20 queens, so you get no points for that, but you get 5 points for each further queen beyond 20. You can mark the grid by placing a dot in each square that contains a queen.

45. A *binary string of length n* is a sequence of n digits, each of which is 0 or 1. The *distance* between two binary strings of the same length is the number of positions in which they disagree; for example, the distance between the strings 01101011 and 00101110 is 3 since they differ in the second, sixth, and eighth positions.

    Find as many binary strings of length 8 as you can, such that the distance between any two of them is at least 3. You get one point per string.

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Team Round

A Build-It-Yourself Table [**150 points**]

An infinite table of nonnegative integers is constructed as follows: in the top row, some number is 1 and all other numbers are 0's; in each subsequent row, every number is the sum of some two of the three closest numbers in the preceding row. An example of such a table is shown below.

$$\begin{array}{ccccccccccc} \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 1 & 3 & 3 & 2 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 2 & 4 & 4 & 6 & 3 & 2 & 0 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

The top row (with the one 1) is called row 0; the next row is row 1; the next row is row 2, and so forth.

Note that the following problems require you to prove the statements for *every* table that can be constructed by the process described above, not just for the example shown.

1. [**10**] Show that any number in row n (for $n > 0$) is at most 2^{n-1}.

2. [**20**] What is the earliest row in which the number 2004 may appear?

3. [**35**] Let
$$S(n,r) = \binom{n-1}{r-1} + \binom{n-1}{r} + \binom{n-1}{r+1} + \cdots + \binom{n-1}{n-1}$$
for all $n, r > 0$, and in particular $S(n,r) = 0$ if $r > n > 0$. Prove that the number in row n of the table, r columns to the left of the 1 in the top row, is at most $S(n,r)$. (**Hint**: First prove that $S(n-1,r-1) + S(n-1,r) = S(n,r)$.)

4. [**25**] Show that the sum of all the numbers in row n is at most $(n+2)2^{n-1}$.

A pair of successive numbers in the same row is called a *switch pair* if one number in the pair is even and the other is odd.

5. [**15**] Prove that the number of switch pairs in row n is at most twice the number of odd numbers in row n.

6. [**20**] Prove that the number of odd numbers in row n is at most twice the number of switch pairs in row $n-1$.

7. [**25**] Prove that the number of switch pairs in row n is at most twice the number of switch pairs in row $n-1$.

Written In The Stars [**125 points**]

Suppose S is a finite set with a binary operation $\star$ — that is, for any elements a, b of S, there is defined an element $a \star b$ of S. It is given that $(a \star b) \star (a \star b) = b \star a$ for all $a, b \in S$.

8. [**20**] Prove that $a \star b = b \star a$ for all $a, b \in S$.

Let T be the set of elements of the form $a \star a$ for $a \in S$.

9. [**15**] If b is any element of T, prove that $b \star b = b$.

Now suppose further that $(a \star b) \star c = a \star (b \star c)$ for all $a, b, c \in S$. (Thus we can write an expression like $a \star b \star c \star d$ without ambiguity.)

10. [**25**] Let a be an element of T. Let the *image* of a be the set of all elements of T that can be represented as $a \star b$ for some $b \in T$. Prove that if c is in the image of a, then $a \star c = c$.

11. [**40**] Prove that there exists an element $a \in T$ such that the equation $a \star b = a$ holds for all $b \in T$.

12. [**25**] Prove that there exists an element $a \in S$ such that the equation $a \star b = a$ holds for all $b \in S$.

Sigma City [**125 points**]

13. [**25**] Let n be a positive odd integer. Prove that

$$\lfloor \log_2 n \rfloor + \lfloor \log_2 (n/3) \rfloor + \lfloor \log_2 (n/5) \rfloor + \lfloor \log_2 (n/7) \rfloor + \cdots + \lfloor \log_2 (n/n) \rfloor = (n-1)/2.$$

Let $\sigma(n)$ denote the sum of the (positive) divisors of n, including 1 and n itself.

14. [**30**] Prove that

$$\sigma(1) + \sigma(2) + \sigma(3) + \cdots + \sigma(n) \leq n^2$$

for every positive integer n.

15. [**30**] Prove that

$$\frac{\sigma(1)}{1} + \frac{\sigma(2)}{2} + \frac{\sigma(3)}{3} + \cdots + \frac{\sigma(n)}{n} \leq 2n$$

for every positive integer n.

16. [**40**] Now suppose again that n is odd. Prove that

$$\sigma(1)\lfloor \log_2 n \rfloor + \sigma(3)\lfloor \log_2 (n/3) \rfloor + \sigma(5)\lfloor \log_2 (n/5) \rfloor + \cdots + \sigma(n)\lfloor \log_2 (n/n) \rfloor < n^2/8.$$

Harvard-MIT Mathematics Tournament

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Individual Round: Algebra Subject Test

1. How many real numbers x are solutions to the following equation?

$$|x-1| = |x-2| + |x-3|$$

2. How many real numbers x are solutions to the following equation?

$$2003^x + 2004^x = 2005^x$$

3. Let x, y, and z be distinct real numbers that sum to 0. Find the maximum possible value of

$$\frac{xy+yz+zx}{x^2+y^2+z^2}.$$

4. If $a, b, c > 0$, what is the smallest possible value of $\left\lfloor \frac{a+b}{c} \right\rfloor + \left\lfloor \frac{b+c}{a} \right\rfloor + \left\lfloor \frac{c+a}{b} \right\rfloor$? (Note that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

5. Ten positive integers are arranged around a circle. Each number is one more than the greatest common divisor of its two neighbors. What is the sum of the ten numbers?

6. Find the sum of the x-coordinates of the distinct points of intersection of the plane curves given by $x^2 = x + y + 4$ and $y^2 = y - 15x + 36$.

7. Let x be a positive real number. Find the maximum possible value of

$$\frac{x^2 + 2 - \sqrt{x^4+4}}{x}.$$

8. Compute

$$\sum_{n=0}^{\infty} \frac{n}{n^4+n^2+1}.$$

9. The number 27,000,001 has exactly four prime factors. Find their sum.

10. Find the sum of the absolute values of the roots of $x^4 - 4x^3 - 4x^2 + 16x - 8 = 0$.

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Individual Round: Geometry Subject Test

1. The volume of a cube (in cubic inches) plus three times the total length of its edges (in inches) is equal to twice its surface area (in square inches). How many inches long is its long diagonal?

2. Let $ABCD$ be a regular tetrahedron with side length 2. The plane parallel to edges AB and CD and lying halfway between them cuts $ABCD$ into two pieces. Find the surface area of one of these pieces.

3. Let $ABCD$ be a rectangle with area 1, and let E lie on side CD. What is the area of the triangle formed by the centroids of triangles ABE, BCE, and ADE?

4. Let XYZ be a triangle with $\angle X = 60^\circ$ and $\angle Y = 45^\circ$. A circle with center P passes through points A and B on side XY, C and D on side YZ, and E and F on side ZX. Suppose $AB = CD = EF$. Find $\angle XPY$ in degrees.

5. A cube with side length 2 is inscribed in a sphere. A second cube, with faces parallel to the first, is inscribed between the sphere and one face of the first cube. What is the length of a side of the smaller cube?

6. A triangular piece of paper of area 1 is folded along a line parallel to one of the sides and pressed flat. What is the minimum possible area of the resulting figure?

7. Let $ABCD$ be a tetrahedron such that edges AB, AC, and AD are mutually perpendicular. Let the areas of triangles ABC, ACD, and ADB be denoted by x, y, and z, respectively. In terms of x, y, and z, find the area of triangle BCD.

8. Let T be a triangle with side lengths 26, 51, and 73. Let S be the set of points inside T which do not lie within a distance of 5 of any side of T. Find the area of S.

9. Let AC be a diameter of a circle ω of radius 1, and let D be the point on AC such that $CD = 1/5$. Let B be the point on ω such that DB is perpendicular to AC, and let E be the midpoint of DB. The line tangent to ω at B intersects line CE at the point X. Compute AX.

10. Let AB be the diameter of a semicircle Γ. Two circles, ω_1 and ω_2, externally tangent to each other and internally tangent to Γ, are tangent to the line AB at P and Q, respectively, and to semicircular arc AB at C and D, respectively, with $AP < AQ$. Suppose F lies on Γ such that $\angle FQB = \angle CQA$ and that $\angle ABF = 80^\circ$. Find $\angle PDQ$ in degrees.

Harvard-MIT Mathematics Tournament

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Individual Round: Combinatorics Subject Test

1. A true-false test has ten questions. If you answer five questions "true" and five "false," your score is guaranteed to be at least four. How many answer keys are there for which this is true?

2. How many nonempty subsets of $\{1, 2, 3, \ldots, 12\}$ have the property that the sum of the largest element and the smallest element is 13?

3. The Red Sox play the Yankees in a best-of-seven series that ends as soon as one team wins four games. Suppose that the probability that the Red Sox win Game n is $\frac{n-1}{6}$. What is the probability that the Red Sox will win the series?

4. In how many ways can 4 purple balls and 4 green balls be placed into a 4×4 grid such that every row and column contains one purple ball and one green ball? Only one ball may be placed in each box, and rotations and reflections of a single configuration are considered different.

5. Doug and Ryan are competing in the 2005 Wiffle Ball Home Run Derby. In each round, each player takes a series of swings. Each swing results in either a home run or an out, and an out ends the series. When Doug swings, the probability that he will hit a home run is $1/3$. When Ryan swings, the probability that he will hit a home run is $1/2$. In one round, what is the probability that Doug will hit more home runs than Ryan hits?

6. Three fair six-sided dice, each numbered 1 through 6, are rolled. What is the probability that the three numbers that come up can form the sides of a triangle?

7. What is the maximum number of bishops that can be placed on an 8×8 chessboard such that at most three bishops lie on any diagonal?

8. Every second, Andrea writes down a random digit uniformly chosen from the set $\{1, 2, 3, 4\}$. She stops when the last two numbers she has written sum to a prime number. What is the probability that the last number she writes down is 1?

9. Eight coins are arranged in a circle heads up. A move consists of flipping over two adjacent coins. How many different sequences of six moves leave the coins alternating heads up and tails up?

10. You start out with a big pile of 3^{2004} cards, with the numbers $1, 2, 3, \ldots, 3^{2004}$ written on them. You arrange the cards into groups of three any way you like; from each group, you keep the card with the largest number and discard the other two. You now again arrange these 3^{2003} remaining cards into groups of three any way you like, and in each group, keep the card with the smallest number and discard the other two. You now have 3^{2002} cards, and you again arrange these into groups of three and keep the largest number in each group. You proceed in this manner, alternating between keeping the largest number and keeping the smallest number in each group, until you have just one card left.

    How many different values are possible for the number on this final card?

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February 19, 2005

Individual Round: Calculus Subject Test

1. Let $f(x) = x^3 + ax + b$, with $a \neq b$, and suppose the tangent lines to the graph of f at $x = a$ and $x = b$ are parallel. Find $f(1)$.

2. A plane curve is parameterized by $x(t) = \int_t^\infty \frac{\cos u}{u}\, du$ and $y(t) = \int_t^\infty \frac{\sin u}{u}\, du$ for $1 \leq t \leq 2$. What is the length of the curve?

3. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function with $\int_0^1 f(x)f'(x)dx = 0$ and $\int_0^1 f(x)^2 f'(x)dx = 18$. What is $\int_0^1 f(x)^4 f'(x)dx$?

4. Let $f : \mathbf{R} \to \mathbf{R}$ be a smooth function such that $f'(x)^2 = f(x)f''(x)$ for all x. Suppose $f(0) = 1$ and $f^{(4)}(0) = 9$. Find all possible values of $f'(0)$.

5. Calculate
$$\lim_{x \to 0^+} \left(x^{x^x} - x^x \right).$$

6. The graph of $r = 2 + \cos 2\theta$ and its reflection over the line $y = x$ bound five regions in the plane. Find the area of the region containing the origin.

7. Two ants, one starting at $(-1, 1)$, the other at $(1, 1)$, walk to the right along the parabola $y = x^2$ such that their midpoint moves along the line $y = 1$ with constant speed 1. When the left ant first hits the line $y = \frac{1}{2}$, what is its speed?

8. If f is a continuous real function such that $f(x-1) + f(x+1) \geq x + f(x)$ for all x, what is the minimum possible value of $\int_1^{2005} f(x)dx$?

9. Compute
$$\sum_{k=0}^{\infty} \frac{4}{(4k)!}.$$

10. Let $f : \mathbf{R} \to \mathbf{R}$ be a smooth function such that $f'(x) = f(1-x)$ for all x and $f(0) = 1$. Find $f(1)$.

Harvard-MIT Mathematics Tournament

February 19, 2005

Individual Round: General Test, Part 1

1. How many real numbers x are solutions to the following equation?

$$|x-1| = |x-2| + |x-3|$$

2. A true-false test has ten questions. If you answer five questions "true" and five "false," your score is guaranteed to be at least four. How many answer keys are there for which this is true?

3. Let $ABCD$ be a regular tetrahedron with side length 2. The plane parallel to edges AB and CD and lying halfway between them cuts $ABCD$ into two pieces. Find the surface area of one of these pieces.

4. Find all real solutions to $x^3 + (x+1)^3 + (x+2)^3 = (x+3)^3$.

5. In how many ways can 4 purple balls and 4 green balls be placed into a 4×4 grid such that every row and column contains one purple ball and one green ball? Only one ball may be placed in each box, and rotations and reflections of a single configuration are considered different.

6. In an election, there are two candidates, A and B, who each have 5 supporters. Each supporter, independent of other supporters, has a $\frac{1}{2}$ probability of voting for his or her candidate and a $\frac{1}{2}$ probability of being lazy and not voting. What is the probability of a tie (which includes the case in which no one votes)?

7. If $a, b, c > 0$, what is the smallest possible value of $\left\lfloor \frac{a+b}{c} \right\rfloor + \left\lfloor \frac{b+c}{a} \right\rfloor + \left\lfloor \frac{c+a}{b} \right\rfloor$? (Note that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

8. Ten positive integers are arranged around a circle. Each number is one more than the greatest common divisor of its two neighbors. What is the sum of the ten numbers?

9. A triangular piece of paper of area 1 is folded along a line parallel to one of the sides and pressed flat. What is the minimum possible area of the resulting figure?

10. What is the smallest integer x larger than 1 such that x^2 ends in the same three digits as x does?

Harvard-MIT Mathematics Tournament

February 19, 2005

Individual Round: General Test, Part 2

1. The volume of a cube (in cubic inches) plus three times the total length of its edges (in inches) is equal to twice its surface area (in square inches). How many inches long is its long diagonal?

2. Find three real numbers $a < b < c$ satisfying:

$$\begin{aligned} a+b+c &= 21/4 \\ 1/a+1/b+1/c &= 21/4 \\ abc &= 1. \end{aligned}$$

3. Working together, Jack and Jill can paint a house in 3 days; Jill and Joe can paint the same house in 4 days; or Joe and Jack can paint the house in 6 days. If Jill, Joe, and Jack all work together, how many days will it take them?

4. In how many ways can 8 people be arranged in a line if Alice and Bob must be next to each other, and Carol must be somewhere behind Dan?

5. You and I play the following game on an 8×8 square grid of boxes: Initially, every box is empty. On your turn, you choose an empty box and draw an X in it; if any of the four adjacent boxes are empty, you mark them with an X as well. (Two boxes are adjacent if they share an edge.) We alternate turns, with you moving first, and whoever draws the last X wins. How many choices do you have for a first move that will enable you to guarantee a win no matter how I play?

6. A cube with side length 2 is inscribed in a sphere. A second cube, with faces parallel to the first, is inscribed between the sphere and one face of the first cube. What is the length of a side of the smaller cube?

7. Three distinct lines are drawn in the plane. Suppose there exist exactly n circles in the plane tangent to all three lines. Find all possible values of n.

8. What is the maximum number of bishops that can be placed on an 8×8 chessboard such that at most three bishops lie on any diagonal?

9. In how many ways can the cells of a 4×4 table be filled in with the digits $1, 2, \ldots, 9$ so that each of the 4-digit numbers formed by the columns is divisible by each of the 4-digit numbers formed by the rows?

10. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. How many positive integers less than 2005 can be expressed in the form $\lfloor x \lfloor x \rfloor \rfloor$ for some positive real x?

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Guts Round

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HARVARD-MIT MATHEMATICS TOURNAMENT, FEBRUARY 19, 2005 — GUTS ROUND

1. **[5]** Find the largest positive integer n such that $1+2+3+\cdots+n^2$ is divisible by $1+2+3+\cdots+n$.

2. **[5]** Let x, y, and z be positive real numbers such that $(x \cdot y) + z = (x+z) \cdot (y+z)$. What is the maximum possible value of xyz?

3. **[5]** Find the sum
$$\frac{2^1}{4^1-1} + \frac{2^2}{4^2-1} + \frac{2^4}{4^4-1} + \frac{2^8}{4^8-1} + \cdots.$$

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4. **[6]** What is the probability that in a randomly chosen arrangement of the numbers and letters in "HMMT2005," one can read either "HMMT" or "2005" from left to right? (For example, in "5HM0M20T," one can read "HMMT.")

5. **[6]** For how many integers n between 1 and 2005, inclusive, is $2 \cdot 6 \cdot 10 \cdots (4n-2)$ divisible by $n!$?

6. **[6]** Let $m \circ n = (m+n)/(mn+4)$. Compute $((\cdots((2005 \circ 2004) \circ 2003) \circ \cdots \circ 1) \circ 0)$.

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7. **[6]** Five people of different heights are standing in line from shortest to tallest. As it happens, the tops of their heads are all collinear; also, for any two successive people, the horizontal distance between them equals the height of the shorter person. If the shortest person is 3 feet tall and the tallest person is 7 feet tall, how tall is the middle person, in feet?

8. **[6]** Let $ABCD$ be a convex quadrilateral inscribed in a circle with shortest side AB. The ratio $[BCD]/[ABD]$ is an integer (where $[XYZ]$ denotes the area of triangle XYZ.) If the lengths of AB, BC, CD, and DA are distinct integers no greater than 10, find the largest possible value of AB.

9. **[6]** Farmer Bill's 1000 animals — ducks, cows, and rabbits — are standing in a circle. In order to feel safe, every duck must either be standing next to at least one cow or between two rabbits. If there are 600 ducks, what is the least number of cows there can be for this to be possible?

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10. [**7**] You are given a set of cards labeled from 1 to 100. You wish to make piles of three cards such that in any pile, the number on one of the cards is the product of the numbers on the other two cards. However, no card can be in more than one pile. What is the maximum number of piles you can form at once?

11. [**7**] The Dingoberry Farm is a 10 mile by 10 mile square, broken up into 1 mile by 1 mile patches. Each patch is farmed either by Farmer Keith or by Farmer Ann. Whenever Ann farms a patch, she also farms all the patches due west of it and all the patches due south of it. Ann puts up a scarecrow on each of her patches that is adjacent to exactly two of Keith's patches (and nowhere else). If Ann farms a total of 30 patches, what is the largest number of scarecrows she could put up?

12. [**7**] Two vertices of a cube are given in space. The locus of points that could be a third vertex of the cube is the union of n circles. Find n.

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HARVARD-MIT MATHEMATICS TOURNAMENT, FEBRUARY 19, 2005 — GUTS ROUND

13. [**7**] Triangle ABC has $AB = 1$, $BC = \sqrt{7}$, and $CA = \sqrt{3}$. Let ℓ_1 be the line through A perpendicular to AB, ℓ_2 the line through B perpendicular to AC, and P the point of intersection of ℓ_1 and ℓ_2. Find PC.

14. [**7**] Three noncollinear points and a line ℓ are given in the plane. Suppose no two of the points lie on a line parallel to ℓ (or ℓ itself). There are exactly n lines perpendicular to ℓ with the following property: the three circles with centers at the given points and tangent to the line all concur at some point. Find all possible values of n.

15. [**7**] Let S be the set of lattice points inside the circle $x^2 + y^2 = 11$. Let M be the greatest area of any triangle with vertices in S. How many triangles with vertices in S have area M?

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16. [**8**] A regular octahedron has a side length of 1. What is the distance between two opposite faces?

17. [**8**] Compute

$$2\sqrt{2\sqrt[3]{2\sqrt[4]{2\sqrt[5]{2\cdots}}}}.$$

18. [**8**] If a, b, and c are random real numbers from 0 to 1, independently and uniformly chosen, what is the average (expected) value of the smallest of a, b, and c?

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19. [**8**] Regular tetrahedron $ABCD$ is projected onto a plane sending A, B, C, and D to A', B', C', and D' respectively. Suppose $A'B'C'D'$ is a convex quadrilateral with $A'B' = B'C$' and $C'D' = D'A'$, and suppose that the area of $A'B'C'D' = 4$. Given these conditions, the set of possible lengths of AB consists of all real numbers in the interval $[a, b)$. Compute b.

20. [**8**] If n is a positive integer, let $s(n)$ denote the sum of the digits of n. We say that n is *zesty* if there exist positive integers x and y greater than 1 such that $xy = n$ and $s(x)s(y) = s(n)$. How many zesty two-digit numbers are there?

21. [**8**] In triangle ABC with altitude AD, $\angle BAC = 45°$, $DB = 3$, and $CD = 2$. Find the area of triangle ABC.

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22. [**9**] Find

$$\{\ln(1+e)\} + \{\ln(1+e^2)\} + \{\ln(1+e^4)\} + \{\ln(1+e^8)\} + \cdots,$$

where $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x.

23. [**9**] The sides of a regular hexagon are trisected, resulting in 18 points, including vertices. These points, starting with a vertex, are numbered clockwise as $A_1, A_2, \ldots, A_{18}$. The line segment A_kA_{k+4} is drawn for $k = 1, 4, 7, 10, 13, 16$, where indices are taken modulo 18. These segments define a region containing the center of the hexagon. Find the ratio of the area of this region to the area of the large hexagon.

24. [**9**] In the base 10 arithmetic problem $HMMT + GUTS = ROUND$, each distinct letter represents a different digit, and leading zeroes are not allowed. What is the maximum possible value of $ROUND$?

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25. **[9]** An ant starts at one vertex of a tetrahedron. Each minute it walks along a random edge to an adjacent vertex. What is the probability that after one hour the ant winds up at the same vertex it started at?

26. **[9]** In triangle ABC, $AC = 3AB$. Let AD bisect angle A with D lying on BC, and let E be the foot of the perpendicular from C to AD. Find $[ABD]/[CDE]$. (Here, $[XYZ]$ denotes the area of triangle XYZ).

27. **[9]** In a chess-playing club, some of the players take lessons from other players. It is possible (but not necessary) for two players both to take lessons from each other. It so happens that for any three distinct members of the club, A, B, and C, exactly one of the following three statements is true: A takes lessons from B; B takes lessons from C; C takes lessons from A. What is the largest number of players there can be?

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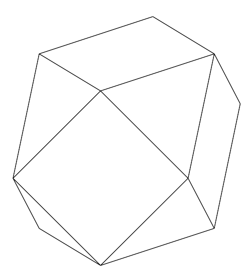
28. **[10]** There are three pairs of real numbers (x_1, y_1), (x_2, y_2), and (x_3, y_3) that satisfy both $x^3 - 3xy^2 = 2005$ and $y^3 - 3x^2y = 2004$. Compute $\left(1 - \frac{x_1}{y_1}\right)\left(1 - \frac{x_2}{y_2}\right)\left(1 - \frac{x_3}{y_3}\right)$.

29. **[10]** Let $n > 0$ be an integer. Each face of a regular tetrahedron is painted in one of n colors (the faces are not necessarily painted different colors.) Suppose there are n^3 possible colorings, where rotations, but not reflections, of the same coloring are considered the same. Find all possible values of n.

30. **[10]** A cuboctahedron is a polyhedron whose faces are squares and equilateral triangles such that two squares and two triangles alternate around each vertex, as shown.



What is the volume of a cuboctahedron of side length 1?

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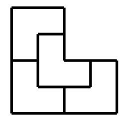
31. **[10]** The L shape made by adjoining three congruent squares can be subdivided into four smaller L shapes.



Each of these can in turn be subdivided, and so forth. If we perform 2005 successive subdivisions, how many of the 4^{2005} L's left at the end will be in the same orientation as the original one?

32. **[10]** Let $a_1 = 3$, and for $n \geq 1$, let $a_{n+1} = (n+1)a_n - n$. Find the smallest $m \geq 2005$ such that $a_{m+1} - 1 \mid a_m^2 - 1$.

33. **[10]** Triangle ABC has incircle ω which touches AB at C_1, BC at A_1, and CA at B_1. Let A_2 be the reflection of A_1 over the midpoint of BC, and define B_2 and C_2 similarly. Let A_3 be the intersection of AA_2 with ω that is closer to A, and define B_3 and C_3 similarly. If $AB = 9$, $BC = 10$, and $CA = 13$, find $[A_3B_3C_3]/[ABC]$. (Here $[XYZ]$ denotes the area of triangle XYZ.)

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34. **[12]** A regular octahedron $ABCDEF$ is given such that AD, BE, and CF are perpendicular. Let G, H, and I lie on edges AB, BC, and CA respectively such that $\frac{AG}{GB} = \frac{BH}{HC} = \frac{CI}{IA} = \rho$. For some choice of $\rho > 1$, GH, HI, and IG are three edges of a regular icosahedron, eight of whose faces are inscribed in the faces of $ABCDEF$. Find ρ.

35. **[12]** Let $p = 2^{24036583} - 1$, the largest prime currently known. For how many positive integers c do each of the quadratics $\pm x^2 \pm px \pm c$ have rational roots?

36. **[12]** One hundred people are in line to see a movie. Each person wants to sit in the front row, which contains one hundred seats, and each has a favorite seat, chosen randomly. They enter the row one at a time from the far right. As they walk, if they reach their favorite seat, they sit, but to avoid stepping over people, if they encounter a person already seated, they sit to that person's right. If the seat furthest to the right is already taken, they sit in a different row. What is the most likely number of people that will get to sit in the first row?

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37. **[15]** Let $a_1, a_2, \ldots, a_{2005}$ be real numbers such that

$$\begin{aligned} a_1 \cdot 1 + a_2 \cdot 2 + a_3 \cdot 3 + \cdots + a_{2005} \cdot 2005 &= 0 \\ a_1 \cdot 1^2 + a_2 \cdot 2^2 + a_3 \cdot 3^2 + \cdots + a_{2005} \cdot 2005^2 &= 0 \\ a_1 \cdot 1^3 + a_2 \cdot 2^3 + a_3 \cdot 3^3 + \cdots + a_{2005} \cdot 2005^3 &= 0 \\ &\vdots \\ a_1 \cdot 1^{2004} + a_2 \cdot 2^{2004} + a_3 \cdot 3^{2004} + \cdots + a_{2005} \cdot 2005^{2004} &= 0 \end{aligned}$$

and

$$a_1 \cdot 1^{2005} + a_2 \cdot 2^{2005} + a_3 \cdot 3^{2005} + \cdots + a_{2005} \cdot 2005^{2005} = 1.$$

What is the value of a_1?

38. **[15]** In how many ways can the set of ordered pairs of integers be colored red and blue such that for all a and b, the points (a, b), $(-1 - b, a + 1)$, and $(1 - b, a - 1)$ are all the same color?

39. **[15]** How many regions of the plane are bounded by the graph of

$$x^6 - x^5 + 3x^4y^2 + 10x^3y^2 + 3x^2y^4 - 5xy^4 + y^6 = 0?$$

. . . .

40. **[18]** In a town of n people, a governing council is elected as follows: each person casts one vote for some person in the town, and anyone that receives at least five votes is elected to council. Let $c(n)$ denote the expected number of people elected to council if everyone votes randomly. Find $\lim_{n\to\infty} c(n)/n$.

41. **[18]** There are 42 stepping stones in a pond, arranged along a circle. You are standing on one of the stones. You would like to jump among the stones so that you move counterclockwise by either 1 stone or 7 stones at each jump. Moreover, you would like to do this in such a way that you visit each stone (except for the starting spot) exactly once before returning to your initial stone for the first time. In how many ways can you do this?

42. **[18]** In how many ways can 6 purple balls and 6 green balls be placed into a 4×4 grid such that every row and column contains two balls of one color and one ball of the other color? Only one ball may be placed in each box, and rotations and reflections of a single configuration are considered different.

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43. Write down an integer N between 0 and 20 inclusive. If at least N teams write down N, your score is N; otherwise it is 0.

44. Write down a set S of positive integers, all greater than 1, whose product is P, such that for each $x \in S$, x is a proper divisor of $(P/x)+1$. Your score is $2n$, where $n = |S|$.

45. A *binary word* is a finite sequence of 0's and 1's. A *square subword* is a subsequence consisting of two identical chunks next to each other. For example, the word 100101011 contains the square subwords 00, 0101 (twice), 1010, and 11.

    Find a long binary word containing a small number of square subwords. Specifically, write down a binary word of any length $n \leq 50$. Your score will be $\max\{0, n - s\}$, where s is the number of occurrences of square subwords. (That is, each different square subword will be counted according to the number of times it appears.)

Harvard-MIT Mathematics Tournament
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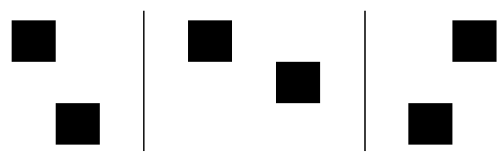
Team Round A

Disconnected Domino Rally [**175**]

On an infinite checkerboard, the union of any two distinct unit squares is called a *(disconnected) domino*. A domino is said to be of *type* (a, b), with $a \leq b$ integers not both zero, if the centers of the two squares are separated by a distance of a in one orthogonal direction and b in the other. (For instance, an ordinary connected domino is of type $(0, 1)$, and a domino of type $(1, 2)$ contains two squares separated by a knight's move.)



Each of the three pairs of squares above forms a domino of type $(1, 2)$.

Two dominoes are said to be *congruent* if they are of the same type. A rectangle is said to be (a, b)-*tileable* if it can be partitioned into dominoes of type (a, b).

1. [**15**] Prove that for any two types of dominoes, there exists a rectangle that can be tiled by dominoes of either type.

2. [**25**] Suppose $0 < a \leq b$ and $4 \nmid mn$. Prove that the number of ways in which an $m \times n$ rectangle can be partitioned into dominoes of type (a, b) is even.

3. [**10**] Show that no rectangle of the form $1 \times k$ or $2 \times n$, where $4 \nmid n$, is $(1, 2)$-tileable.

4. [**35**] Show that all other rectangles of even area are $(1, 2)$-tileable.

5. [**25**] Show that for b even, there exists some M such that for every $n > M$, a $2b \times n$ rectangle is $(1, b)$-tileable.

6. [**40**] Show that for b even, there exists some M such that for every $m, n > M$ with mn even, an $m \times n$ rectangle is $(1, b)$-tileable.

7. [**25**] Prove that neither of the previous two problems holds if b is odd.

An Interlude — Discovering One's Roots [**100**]

A *kth root of unity* is any complex number ω such that $\omega^k = 1$. You may use the following facts: if $\omega \neq 1$, then

$$1 + \omega + \omega^2 + \cdots + \omega^{k-1} = 0,$$

and if $1, \omega, \ldots, \omega^{k-1}$ are distinct, then

$$(x^k - 1) = (x - 1)(x - \omega)(x - \omega^2) \cdots (x - \omega^{k-1}).$$

8. **[25]** Suppose x is a fifth root of unity. Find, in radical form, all possible values of

$$2x + \frac{1}{1+x} + \frac{x}{1+x^2} + \frac{x^2}{1+x^3} + \frac{x^3}{1+x^4}.$$

9. **[25]** Let $A_1A_2\ldots A_k$ be a regular k-gon inscribed in a circle of radius 1, and let P be a point lying on or inside the circumcircle. Find the maximum possible value of $(PA_1)(PA_2)\cdots(PA_k)$.

10. **[25]** Let P be a regular k-gon inscribed in a circle of radius 1. Find the sum of the squares of the lengths of all the sides and diagonals of P.

11. **[25]** Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with real coefficients, $a_n \neq 0$. Suppose every root of P is a root of unity, but $P(1) \neq 0$. Show that the coefficients of P are symmetric; that is, show that $a_n = a_0, a_{n-1} = a_1, \ldots$

Early Re-tile-ment [125]

Let $S = \{s_0, \ldots, s_n\}$ be a finite set of integers, and define $S + k = \{s_0 + k, \ldots, s_n + k\}$. We say that two sets S and T are *equivalent*, written $S \sim T$, if $T = S + k$ for some k. Given a (possibly infinite) set of integers A, we say that S *tiles* A if A can be partitioned into subsets equivalent to S. Such a partition is called a *tiling* of A by S.

12. **[20]** Suppose the elements of A are either bounded below or bounded above. Show that if S tiles A, then it does so uniquely, i.e., there is a unique tiling of A by S.

13. **[35]** Let B be a set of integers either bounded below or bounded above. Then show that if S tiles all other integers $\mathbf{Z}\backslash B$, then S tiles all integers $\mathbf{Z}$.

14. **[35]** Suppose S tiles the natural numbers $\mathbf{N}$. Show that S tiles the set $\{1, 2, \ldots, k\}$ for some positive integer k.

15. **[35]** Suppose S tiles $\mathbf{N}$. Show that S is symmetric; that is, if $-S = \{-s_n, \ldots, -s_0\}$, show that $S \sim -S$.

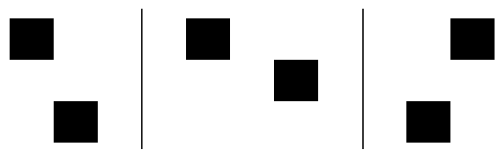
Harvard-MIT Mathematics Tournament

February 19, 2005

Team Round B

Disconnected Domino Rally [**150**]

On an infinite checkerboard, the union of any two distinct unit squares is called a *(disconnected) domino*. A domino is said to be of *type* (a, b), with $a \leq b$ integers not both zero, if the centers of the two squares are separated by a distance of a in one orthogonal direction and b in the other. (For instance, an ordinary connected domino is of type $(0, 1)$, and a domino of type $(1, 2)$ contains two squares separated by a knight's move.)



Each of the three pairs of squares above forms a domino of type $(1, 2)$.

Two dominoes are said to be *congruent* if they are of the same type. A rectangle is said to be (a, b)-*tileable* if it can be partitioned into dominoes of type (a, b).

1. [**15**] Let $0 < m \leq n$ be integers. How many different (i.e., noncongruent) dominoes can be formed by choosing two squares of an $m \times n$ array?

2. [**10**] What are the dimensions of the rectangle of smallest area that is (a, b)-tileable?

3. [**20**] Prove that every (a, b)-tileable rectangle contains a rectangle of these dimensions.

4. [**30**] Prove that an $m \times n$ rectangle is (b, b)-tileable if and only if $2b \mid m$ and $2b \mid n$.

5. [**35**] Prove that an $m \times n$ rectangle is $(0, b)$-tileable if and only if $2b \mid m$ or $2b \mid n$.

6. [**40**] Let k be an integer such that $k \mid a$ and $k \mid b$. Prove that if an $m \times n$ rectangle is (a, b)-tileable, then $2k \mid m$ or $2k \mid n$.

An Interlude — Discovering One's Roots [**100**]

A *kth root of unity* is any complex number ω such that $\omega^k = 1$.

7. [**15**] Find a real, irreducible quartic polynomial with leading coefficient 1 whose roots are all twelfth roots of unity.

8. [**25**] Let x and y be two kth roots of unity. Prove that $(x + y)^k$ is real.

9. [**30**] Let x and y be two distinct roots of unity. Prove that $x + y$ is also a root of unity if and only if $\frac{y}{x}$ is a cube root of unity.

10. [**30**] Let x, y, and z be three roots of unity. Prove that $x + y + z$ is also a root of unity if and only if $x + y = 0$, $y + z = 0$, or $z + x = 0$.

Early Re-tile-ment [**150**]

Let $S = \{s_0, \ldots, s_n\}$ be a finite set of integers, and define $S + k = \{s_0 + k, \ldots, s_n + k\}$. We say that S and T are *equivalent*, written $S \sim T$, if $T = S + k$ for some k. Given a (possibly infinite) set of integers A, we say that S *tiles* A if A can be partitioned into subsets equivalent to S. Such a partition is called a *tiling* of A by S.

11. [**20**] Find all sets S with minimum element 1 that tile $A = \{1, \ldots, 12\}$.

12. [**35**] Let A be a finite set with more than one element. Prove that the number of nonequivalent sets S which tile A is always even.

13. [**25**] Exhibit a set S which tiles the integers $\mathbf{Z}$ but not the natural numbers $\mathbf{N}$.

14. [**30**] Suppose that S tiles the set of all integer cubes. Prove that S has only one element.

15. [**40**] Suppose that S tiles the set of odd prime numbers. Prove that S has only one element.

IX$^{\text{th}}$ Annual Harvard-MIT Mathematics Tournament

Saturday 25 February 2006

Individual Round: Algebra Test

1. Larry can swim from Harvard to MIT (with the current of the Charles River) in 40 minutes, or back (against the current) in 45 minutes. How long does it take him to *row* from Harvard to MIT, if he rows the return trip in 15 minutes? (Assume that the speed of the current and Larry's swimming and rowing speeds relative to the current are all constant.) Express your answer in the format mm:ss.

2. Find all real solutions (x, y) of the system $x^2 + y = 12 = y^2 + x$.

3. The train schedule in Hummut is hopelessly unreliable. Train A will enter Intersection X from the west at a random time between 9:00 am and 2:30 pm; each moment in that interval is equally likely. Train B will enter the same intersection from the north at a random time between 9:30 am and 12:30 pm, independent of Train A; again, each moment in the interval is equally likely. If each train takes 45 minutes to clear the intersection, what is the probability of a collision today?

4. Let $a_1, a_2, \ldots$ be a sequence defined by $a_1 = a_2 = 1$ and $a_{n+2} = a_{n+1} + a_n$ for $n \geq 1$. Find
$$\sum_{n=1}^{\infty} \frac{a_n}{4^{n+1}}.$$

5. Tim has a working analog 12-hour clock with two hands that run continuously (instead of, say, jumping on the minute). He also has a clock that runs really slow—at half the correct rate, to be exact. At noon one day, both clocks happen to show the exact time. At any given instant, the hands on each clock form an angle between 0° and 180° inclusive. At how many times during that day are the angles on the two clocks equal?

6. Let a, b, c be the roots of $x^3 - 9x^2 + 11x - 1 = 0$, and let $s = \sqrt{a} + \sqrt{b} + \sqrt{c}$. Find $s^4 - 18s^2 - 8s$.

7. Let
$$f(x) = x^4 - 6x^3 + 26x^2 - 46x + 65.$$
Let the roots of $f(x)$ be $a_k + ib_k$ for $k = 1, 2, 3, 4$. Given that the a_k, b_k are all integers, find $|b_1| + |b_2| + |b_3| + |b_4|$.

8. Solve for all complex numbers z such that $z^4 + 4z^2 + 6 = z$.

9. Compute the value of the infinite series
$$\sum_{n=2}^{\infty} \frac{n^4 + 3n^2 + 10n + 10}{2^n \cdot (n^4 + 4)}$$

10. Determine the maximum value attained by
$$\frac{x^4 - x^2}{x^6 + 2x^3 - 1}$$
over real numbers $x > 1$.

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Individual Round: Geometry Test

1. Octagon $ABCDEFGH$ is equiangular. Given that $AB = 1$, $BC = 2$, $CD = 3$, $DE = 4$, and $EF = FG = 2$, compute the perimeter of the octagon.

2. Suppose ABC is a scalene right triangle, and P is the point on hypotenuse $\overline{AC}$ such that $\angle ABP = 45°$. Given that $AP = 1$ and $CP = 2$, compute the area of ABC.

3. Let A, B, C, and D be points on a circle such that $AB = 11$ and $CD = 19$. Point P is on segment AB with $AP = 6$, and Q is on segment CD with $CQ = 7$. The line through P and Q intersects the circle at X and Y. If $PQ = 27$, find XY.

4. Let ABC be a triangle such that $AB = 2, CA = 3$, and $BC = 4$. A semicircle with its diameter on $\overline{BC}$ is tangent to $\overline{AB}$ and $\overline{AC}$. Compute the area of the semicircle.

5. Triangle ABC has side lengths $AB = 2\sqrt{5}$, $BC = 1$, and $CA = 5$. Point D is on side AC such that $CD = 1$, and F is a point such that $BF = 2$ and $CF = 3$. Let E be the intersection of lines AB and DF. Find the area of $CDEB$.

6. A circle of radius t is tangent to the hypotenuse, the incircle, and one leg of an isosceles right triangle with inradius $r = 1 + \sin\frac{\pi}{8}$. Find rt.

7. Suppose $ABCD$ is an isosceles trapezoid in which $\overline{AB} \parallel \overline{CD}$. Two mutually externally tangent circles ω_1 and ω_2 are inscribed in $ABCD$ such that ω_1 is tangent to $\overline{AB}, \overline{BC}$, and $\overline{CD}$ while ω_2 is tangent to $\overline{AB}, \overline{DA}$, and $\overline{CD}$. Given that $AB = 1, CD = 6$, compute the radius of either circle.

8. Triangle ABC has a right angle at B. Point D lies on side BC such that $3\angle BAD = \angle BAC$. Given $AC = 2$ and $CD = 1$, compute BD.

9. Four spheres, each of radius r, lie inside a regular tetrahedron with side length 1 such that each sphere is tangent to three faces of the tetrahedron and to the other three spheres. Find r.

10. Triangle ABC has side lengths $AB = 65$, $BC = 33$, and $AC = 56$. Find the radius of the circle tangent to sides AC and BC and to the circumcircle of triangle ABC.

IX$^{\text{th}}$ Annual Harvard-MIT Mathematics Tournament

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Individual Round: Combinatorics Test

1. Vernonia High School has 85 seniors, each of whom plays on at least one of the school's three varsity sports teams: football, baseball, and lacrosse. It so happens that 74 are on the football team; 26 are on the baseball team; 17 are on both the football and lacrosse teams; 18 are on both the baseball and football teams; and 13 are on both the baseball and lacrosse teams. Compute the number of seniors playing all three sports, given that twice this number are members of the lacrosse team.

2. Compute
$$\sum_{n_{60}=0}^{2} \sum_{n_{59}=0}^{n_{60}} \cdots \sum_{n_2=0}^{n_3} \sum_{n_1=0}^{n_2} \sum_{n_0=0}^{n_1} 1.$$

3. A moth starts at vertex A of a certain cube and is trying to get to vertex B, which is opposite A, in five or fewer "steps," where a step consists in traveling along an edge from one vertex to another. The moth will stop as soon as it reaches B. How many ways can the moth achieve its objective?

4. A dot is marked at each vertex of a triangle ABC. Then, 2, 3, and 7 more dots are marked on the sides AB, BC, and CA, respectively. How many triangles have their vertices at these dots?

5. Fifteen freshmen are sitting in a circle around a table, but the course assistant (who remains standing) has made only six copies of today's handout. No freshman should get more than one handout, and any freshman who does not get one should be able to read a neighbor's. If the freshmen are distinguishable but the handouts are not, how many ways are there to distribute the six handouts subject to the above conditions?

6. For how many ordered triplets (a, b, c) of positive integers less than 10 is the product $a \times b \times c$ divisible by 20?

7. Let n be a positive integer, and let Pushover be a game played by two players, standing squarely facing each other, pushing each other, where the first person to lose balance loses. At the HMPT, 2^{n+1} competitors, numbered 1 through 2^{n+1} clockwise, stand in a circle. They are equals in Pushover: whenever two of them face off, each has a 50% probability of victory. The tournament unfolds in $n+1$ rounds. In each round, the referee randomly chooses one of the surviving players, and the players pair off going clockwise, starting from the chosen one. Each pair faces off in Pushover, and the losers leave the circle. What is the probability that players 1 and 2^n face each other in the last round? Express your answer in terms of n.

8. In how many ways can we enter numbers from the set $\{1, 2, 3, 4\}$ into a 4×4 array so that all of the following conditions hold?

   (a) Each row contains all four numbers.

   (b) Each column contains all four numbers.

   (c) Each "quadrant" contains all four numbers. (The quadrants are the four corner 2×2 squares.)

9. Eight celebrities meet at a party. It so happens that each celebrity shakes hands with exactly two others. A fan makes a list of all unordered pairs of celebrities who shook hands with each other. If order does not matter, how many different lists are possible?

10. Somewhere in the universe, n students are taking a 10-question math competition. Their collective performance is called *laughable* if, for some pair of questions, there exist 57 students such that either all of them answered both questions correctly or none of them answered both questions correctly. Compute the smallest n such that the performance is necessarily laughable.

IX$^{\text{th}}$ Annual Harvard-MIT Mathematics Tournament

Saturday 25 February 2006

Individual Round: Calculus Test

1. A nonzero polynomial $f(x)$ with real coefficients has the property that $f(x) = f'(x)f''(x)$. What is the leading coefficient of $f(x)$?

2. Compute $\lim_{x\to 0} \dfrac{e^{x\cos x} - 1 - x}{\sin(x^2)}$.

3. At time 0, an ant is at $(1,0)$ and a spider is at $(-1,0)$. The ant starts walking counterclockwise along the unit circle, and the spider starts creeping to the right along the x-axis. It so happens that the ant's horizontal speed is always half the spider's. What will the shortest distance ever between the ant and the spider be?

4. Compute $\displaystyle\sum_{k=1}^{\infty} \frac{k^4}{k!}$.

5. Compute $\displaystyle\int_0^1 \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$.

6. A triangle with vertices at $(1003,0)$, $(1004,3)$, and $(1005,1)$ in the xy-plane is revolved all the way around the y-axis. Find the volume of the solid thus obtained.

7. Find all positive real numbers c such that the graph of $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 - cx$ has the property that the circle of curvature at any local extremum is centered at a point on the x-axis.

8. Compute $\displaystyle\int_0^{\pi/3} x\tan^2(x)dx$.

9. Compute the sum of all real numbers x such that

$$2x^6 - 3x^5 + 3x^4 + x^3 - 3x^2 + 3x - 1 = 0.$$

10. Suppose f and g are differentiable functions such that

$$xg(f(x))f'(g(x))g'(x) = f(g(x))g'(f(x))f'(x)$$

for all real x. Moreover, f is nonnegative and g is positive. Furthermore,

$$\int_0^a f(g(x))dx = 1 - \frac{e^{-2a}}{2}$$

for all reals a. Given that $g(f(0)) = 1$, compute the value of $g(f(4))$.

IX^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 25 February 2006

Individual Round: General Test, Part 1

1. How many positive integers x are there such that $3x$ has 3 digits and $4x$ has four digits?

2. What is the probability that two cards randomly selected (without replacement) from a standard 52-card deck are neither of the same value nor the same suit?

3. A square and an equaliteral triangle together have the property that the area of each is the perimeter of the other. Find the square's area.

4. Find
$$\frac{\sqrt{31+\sqrt{31+\sqrt{31+\ldots}}}}{\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}.$$

5. In the plane, what is the length of the shortest path from $(-2,0)$ to $(2,0)$ that avoids the interior of the unit circle (i.e., circle of radius 1) centered at the origin?

6. Six celebrities meet at a party. It so happens that each celebrity shakes hands with exactly two others. A fan makes a list of all unordered pairs of celebrities who shook hands with each other. If order does not matter, how many different lists are possible?

7. The train schedule in Hummut is hopelessly unreliable. Train A will enter Intersection X from the west at a random time between 9:00 am and 2:30 pm; each moment in that interval is equally likely. Train B will enter the same intersection from the north at a random time between 9:30 am and 12:30 pm, independent of Train A; again, each moment in the interval is equally likely. If each train takes 45 minutes to clear the intersection, what is the probability of a collision today?

8. A dot is marked at each vertex of a triangle ABC. Then, 2, 3, and 7 more dots are marked on the sides AB, BC, and CA, respectively. How many triangles have their vertices at these dots?

9. Take a unit sphere S, i.e., a sphere with radius 1. Circumscribe a cube C about S, and inscribe a cube D in S, so that every edge of cube C is parallel to some edge of cube D. What is the shortest possible distance from a point on a face of C to a point on a face of D?

10. A positive integer n is called "flippant" if n does not end in 0 (when written in decimal notation) and, moreover, n and the number obtained by reversing the digits of n are both divisible by 7. How many flippant integers are there between 10 and 1000?

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Individual Round: General Test, Part 2

1. Larry can swim from Harvard to MIT (with the current of the Charles River) in 40 minutes, or back (against the current) in 45 minutes. How long does it take him to *row* from Harvard to MIT, if he rows the return trip in 15 minutes? (Assume that the speed of the current and Larry's swimming and rowing speeds relative to the current are all constant.) Express your answer in the format mm:ss.

2. Find
$$\frac{2^2}{2^2-1} \cdot \frac{3^2}{3^2-1} \cdot \frac{4^2}{4^2-1} \cdot \cdots \cdot \frac{2006^2}{2006^2-1}.$$

3. Let C be the unit circle. Four distinct, smaller congruent circles C_1, C_2, C_3, C_4 are internally tangent to C such that C_i is externally tangent to C_{i-1} and C_{i+1} for $i = 1, \ldots, 4$ where C_5 denotes C_1 and C_0 represents C_4. Compute the radius of C_1.

4. Vernonia High School has 85 seniors, each of whom plays on at least one of the school's three varsity sports teams: football, baseball, and lacrosse. It so happens that 74 are on the football team; 26 are on the baseball team; 17 are on both the football and lacrosse teams; 18 are on both the baseball and football teams; and 13 are on both the baseball and lacrosse teams. Compute the number of seniors playing all three sports, given that twice this number are members of the lacrosse team.

5. If a, b are nonzero real numbers such that $a^2 + b^2 = 8ab$, find the value of $\left|\frac{a+b}{a-b}\right|$.

6. Octagon $ABCDEFGH$ is equiangular. Given that $AB = 1$, $BC = 2$, $CD = 3$, $DE = 4$, and $EF = FG = 2$, compute the perimeter of the octagon.

7. What is the smallest positive integer n such that n^2 and $(n+1)^2$ both contain the digit 7 but $(n+2)^2$ does not?

8. Six people, all of different weights, are trying to build a human pyramid: that is, they get into the formation

   A

   B C

   D E F

   We say that someone not in the bottom row is "supported by" each of the two closest people beneath her or him. How many different pyramids are possible, if nobody can be supported by anybody of lower weight?

9. Tim has a working analog 12-hour clock with two hands that run continuously (instead of, say, jumping on the minute). He also has a clock that runs really slow—at half the correct rate, to be exact. At noon one day, both clocks happen to show the exact time. At any given instant, the hands on each clock form an angle between 0° and 180° inclusive. At how many times during that day are the angles on the two clocks equal?

10. Fifteen freshmen are sitting in a circle around a table, but the course assistant (who remains standing) has made only six copies of today's handout. No freshman should get more than one handout, and any freshman who does not get one should be able to read a neighbor's. If the freshmen are distinguishable but the handouts are not, how many ways are there to distribute the six handouts subject to the above conditions?

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Guts Round

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1. **[5]** A bear walks one mile south, one mile east, and one mile north, only to find itself where it started. Another bear, more energetic than the first, walks two miles south, two miles east, and two miles north, only to find itself where it started. However, the bears are *not* white and did *not* start at the north pole. At most how many miles apart, to the nearest .001 mile, are the two bears' starting points?

2. **[5]** Compute the positive integer less than 1000 which has exactly 29 positive proper divisors. (Here we refer to positive integer divisors other than the number itself.)

3. **[5]** At a nursey, 2006 babies sit in a circle. Suddenly each baby pokes the baby immediately to either its left or its right, with equal probability. What is the expected number of unpoked babies?

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4. **[6]** Ann and Anne are in bumper cars starting 50 meters apart. Each one approaches the other at a constant ground speed of 10 km/hr. A fly starts at Ann, flies to Anne, then back to Ann, and so on, back and forth until it gets crushed when the two bumper cars collide. When going from Ann to Anne, the fly flies at 20 km/hr; when going in the opposite direction the fly flies at 30 km/hr (thanks to a breeze). How many meters does the fly fly?

5. **[6]** Find the number of solutions in positive integers $(k; a_1, a_2, \ldots, a_k; b_1, b_2, \ldots, b_k)$ to the equation $a_1(b_1) + a_2(b_1 + b_2) + \cdots + a_k(b_1 + b_2 + \cdots + b_k) = 7$.

6. **[6]** Suppose ABC is a triangle such that $AB = 13, BC = 15$, and $CA = 14$. Say D is the midpoint of $\overline{BC}$, E is the midpoint of $\overline{AD}$, F is the midpoint of $\overline{BE}$, and G is the midpoint of $\overline{DF}$. Compute the area of triangle EFG.

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7. **[6]** Find all real numbers x such that $x^2 + \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x}{3} \right\rfloor = 10$.

8. **[6]** How many ways are there to label the faces of a regular octahedron with the integers 1–8, using each exactly once, so that any two faces that share an edge have numbers that are relatively prime? Physically realizable rotations are considered indistinguishable, but physically unrealizable reflections are considered different.

9. **[6]** Four unit circles are centered at the vertices of a unit square, one circle at each vertex. What is the area of the region common to all four circles?

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10. **[7]** Let $f(x) = x^2 - 2x$. How many distinct real numbers c satisfy $f(f(f(f(c)))) = 3$?

11. **[7]** Find all positive integers $n > 1$ for which $\frac{n^2+7n+136}{n-1}$ is the square of a positive integer.

12. **[7]** For each positive integer n let S_n denote the set $\{1, 2, 3, \ldots, n\}$. Compute the number of triples of subsets A, B, C of S_{2006} (not necessarily nonempty or proper) such that A is a subset of B and $S_{2006} - A$ is a subset of C.

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The problems in this batch all depend on each other. If you solve them correctly, you will produce a triple of mutually consistent answers. There is only one such triple. Your score will be determined by how many of your answers match that triple.

13. **[7]** Let Z be as in problem 15. Let X be the greatest integer such that $|XZ| \leq 5$. Find X.

14. **[7]** Let X be as in problem 13. Let Y be the number of ways to order X crimson flowers, X scarlet flowers, and X vermillion flowers in a row so that no two flowers of the same hue are adjacent. (Flowers of the same hue are mutually indistinguishable.) Find Y.

15. **[7]** Let Y be as in problem 14. Find the maximum Z such that three circles of radius $\sqrt{Z}$ can simultaneously fit inside an equilateral triangle of area Y without overlapping each other.

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16. **[8]** A sequence $a_1, a_2, a_3, \ldots$ of positive reals satisfies $a_{n+1} = \sqrt{\frac{1+a_n}{2}}$. Determine all a_1 such that $a_i = \frac{\sqrt{6}+\sqrt{2}}{4}$ for some positive integer i.

17. **[8]** Begining at a vertex, an ant is crawls between the vertices of a regular octahedron. After reaching a vertex, it randomly picks a neighboring vertex (sharing an edge) and walks to that vertex along the adjoining edge (with all possibilities equally likely.) What is the probability that after walking along 2006 edges, the ant returns to the vertex where it began?

18. **[8]** Cyclic quadrilateral $ABCD$ has side lengths $AB = 1, BC = 2, CD = 3$ and $DA = 4$. Points P and Q are the midpoints of $\overline{BC}$ and $\overline{DA}$. Compute PQ^2.

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19. **[8]** Let ABC be a triangle with $AB = 2, CA = 3, BC = 4$. Let D be the point diametrically opposite A on the circumcircle of ABC, and let E lie on line AD such that D is the midpoint of $\overline{AE}$. Line l passes through E perpendicular to $\overline{AE}$, and F and G are the intersections of the extensions of $\overline{AB}$ and $\overline{AC}$ with l. Compute FG.

20. **[8]** Compute the number of real solutions (x, y, z, w) to the system of equations:

$$x = z + w + zwx \qquad z = x + y + xyz$$
$$y = w + x + wxy \qquad w = y + z + yzw$$

21. **[8]** Find the smallest positive integer k such that $z^{10} + z^9 + z^6 + z^5 + z^4 + z + 1$ divides $z^k - 1$.

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22. **[9]** Let $f(x)$ be a degree 2006 polynomial with complex roots $c_1, c_2, \ldots, c_{2006}$, such that the set $\{|c_1|, |c_2|, \ldots, |c_{2006}|\}$ consists of exactly 1006 distinct values. What is the minimum number of real roots of $f(x)$?

23. **[9]** Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers defined by $a_0 = 21, a_1 = 35$, and $a_{n+2} = 4a_{n+1} - 4a_n + n^2$ for $n \geq 2$. Compute the remainder obtained when a_{2006} is divided by 100.

24. **[9]** Two 18-24-30 triangles in the plane share the same circumcircle as well as the same incircle. What's the area of the region common to both the triangles?

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25. **[9]** Points A, C, and B lie on a line in that order such that $AC = 4$ and $BC = 2$. Circles ω_1, ω_2, and ω_3 have $\overline{BC}, \overline{AC}$, and $\overline{AB}$ as diameters. Circle Γ is externally tangent to ω_1 and ω_2 at D and E respectively, and is internally tangent to ω_3. Compute the circumradius of triangle CDE.

26. **[9]** Let $a \geq b \geq c$ be real numbers such that

$$\begin{aligned} a^2bc + ab^2c + abc^2 + 8 &= a + b + c \\ a^2b + a^2c + b^2c + b^2a + c^2a + c^2b + 3abc &= -4 \\ a^2b^2c + ab^2c^2 + a^2bc^2 &= 2 + ab + bc + ca \end{aligned}$$

If $a + b + c > 0$, then compute the integer nearest to a^5.

27. **[9]** Let N denote the number of subsets of $\{1, 2, 3, \ldots, 100\}$ that contain more prime numbers than multiples of 4. Compute the largest integer k such that 2^k divides N.

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28. **[10]** A pebble is shaped as the intersection of a cube of side length 1 with the solid sphere tangent to all of the cube's edges. What is the surface area of this pebble?

29. **[10]** Find the area in the first quadrant bounded by the hyperbola $x^2 - y^2 = 1$, the x-axis, and the line $3x = 4y$.

30. **[10]** ABC is an acute triangle with incircle ω. ω is tangent to sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at D, E, and F respectively. P is a point on the altitude from A such that Γ, the circle with diameter $\overline{AP}$, is tangent to ω. Γ intersects $\overline{AC}$ and $\overline{AB}$ at X and Y respectively. Given $XY = 8$, $AE = 15$, and that the radius of Γ is 5, compute $BD \cdot DC$.

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The problems in this batch all depend on each other. If you solve them correctly, you will produce a triple of mutually consistent answers. There is only one such triple. Your score will be determined by how many of your answers match that triple.

31. **[10]** Let A be as in problem 33. Let W be the sum of all positive integers that divide A. Find W.

32. **[10]** In the alphametic $WE \times EYE = SCENE$, each different letter stands for a different digit, and no word begins with a 0. The W in this problem has the same value as the W in problem 31. Find S.

33. **[10]** Let W, S be as in problem 32. Let A be the least positive integer such that an acute triangle with side lengths S, A, and W exists. Find A.

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34. **[12]** In bridge, a standard 52-card deck is dealt in the usual way to 4 players. By convention, each hand is assigned a number of "points" based on the formula

$$4 \times (\#\text{ A's}) + 3 \times (\#\text{ K's}) + 2 \times (\#\text{ Q's}) + 1 \times (\#\text{ J's}).$$

Given that a particular hand has exactly 4 cards that are A, K, Q, or J, find the probability that its point value is 13 or higher.

35. **[12]** A sequence is defined by $A_0 = 0, A_1 = 1, A_2 = 2$, and, for integers $n \geq 3$,

$$A_n = \frac{A_{n-1} + A_{n-2} + A_{n-3}}{3} + \frac{1}{n^4 - n^2}$$

Compute $\lim_{N \to \infty} A_N$.

36. **[12]** Four points are independently chosen uniformly at random from the interior of a regular dodecahedron. What is the probability that they form a tetrahedron whose interior contains the dodecahedron's center?

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37. **[15]** Compute $\displaystyle\sum_{n=1}^{\infty} \frac{2n+5}{2^n \cdot (n^3+7n^2+14n+8)}$

38. **[15]** Suppose ABC is a triangle with incircle ω, and ω is tangent to $\overline{BC}$ and $\overline{CA}$ at D and E respectively. The bisectors of $\angle A$ and $\angle B$ intersect line DE at F and G respectively, such that $BF = 1$ and $FG = GA = 6$. Compute the radius of ω.

39. **[15]** A *fat coin* is one which, when tossed, has a 2/5 probability of being heads, 2/5 of being tails, and 1/5 of landing on its edge. Mr. Fat starts at 0 on the real line. Every minute, he tosses a fat coin. If it's heads, he moves left, decreasing his coordinate by 1; if it's tails, he moves right, increasing his coordinate by 1. If the coin lands on its edge, he moves back to 0. If Mr. Fat does this *ad infinitum*, what fraction of his time will he spend at 0?

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40. **[18]** Compute $\displaystyle\sum_{k=1}^{\infty} \frac{3k+1}{2k^3+k^2} \cdot (-1)^{k+1}$.

41. **[18]** Let Γ denote the circumcircle of triangle ABC. Point D is on $\overline{AB}$ such that $\overline{CD}$ bisects $\angle ACB$. Points P and Q are on Γ such that $\overline{PQ}$ passes through D and is perpendicular to $\overline{CD}$. Compute PQ, given that $BC = 20, CA = 80, AB = 65$.

42. **[18]** Suppose hypothetically that a certain, very corrupt political entity in a universe holds an election with two candidates, say A and B. A total of 5,825,043 votes are cast, but, in a sudden rainstorm, all the ballots get soaked. Undaunted, the election officials decide to guess what the ballots say. Each ballot has a 51% chance of being deemed a vote for A, and a 49% chance of being deemed a vote for B. The probability that B will win is 10^{-X}. What is X *rounded to the nearest* 10?

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43. Write down at least one, and up to ten, different 3-digit prime numbers. If you somehow fail to do this, we will ignore your submission for this problem. Otherwise, you're entered into a game with other teams. In this game, you start with 10 points, and each number you write down is like a bet: if no one else writes that number, you gain 1 point, but if anyone else writes that number, you lose 1 point. Thus, your score on this problem can be anything from 0 to 20.

44. On the Euclidean plane are given 14 points:

$A = (0, 428)$	$B = (9, 85)$	$C = (42, 865)$	$D = (192, 875)$
$E = (193, 219)$	$F = (204, 108)$	$G = (292, 219)$	$H = (316, 378)$
$I = (375, 688)$	$J = (597, 498)$	$K = (679, 766)$	$L = (739, 641)$
	$M = (772, 307)$	$N = (793, 0)$	

A fly starts at A, visits all the other points, and comes back to A in such a way as to minimize the total distance covered. What path did the fly take? Give the names of the points it visits in order. Your score will be

$$20 + \lfloor \text{the optimal distance} \rfloor - \lfloor \text{your distance} \rfloor$$

or 0, whichever is greater.

45. On your answer sheet, *clearly* mark at least seven points, as long as

(i) No three are collinear.

(ii) No seven form a convex heptagon.

Please do not cross out any points; erase if you can do so neatly. If the graders deem that your paper is too messy, or if they determine that you violated one of those conditions, your submission for this problem will be disqualified. Otherwise, your score will be the number of points you marked minus 6, even if you actually violated one of the conditions but were able to fool the graders.

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Team Round A

Mobotics [120]

Spring is finally here in Cambridge, and it's time to mow our lawn. For the purpose of these problems, our lawn consists of little *clumps* of grass arranged at the points of a certain grid (to be specified later). Our machinery consists of a fleet of identical *mowbots* (or "mobots" for short). A mobot is a lawn-mowing machine. To mow our lawn, we begin by choosing a *formation*: we place as many mobots as we want at various clumps of grass and orient each mobot's head in a certain direction. At the blow of a whistle, each mobot starts moving in the direction we've chosen, mowing every clump of grass in its path (including the clump it starts on) until it goes off the lawn.

Because the spring is so young, our lawn is rather delicate. Consequently, we want to make sure that every clump of grass is mowed once and only once. We will not consider formations that do not meet this criterion.

One more thing: two formations are considered "different" if there exists a clump of grass for which either (1) for exactly one of the formations does a mobot starts on that clump, or (2) there are mobots starting on this clump for both the formations, but they're oriented in different directions.

1. **[15]** For this problem, our lawn consists of a row of n clumps of grass. This row runs in an east-west direction. In our formation, each mobot may be oriented toward the north, south, east, or west. One example of an allowable formation if $n = 6$ is symbolized below:

$$\cdot \quad \cdot \leftarrow \quad \uparrow \quad \uparrow \quad \downarrow$$

(The mobot on the third clump will move westward, mowing the first three clumps. Each of the last three clumps is mowed by a different mobot.) Here's another allowable formation for $n = 6$, considered different from the first:

$$\cdot \quad \cdot \leftarrow \quad \uparrow \quad \uparrow \quad \rightarrow$$

Compute the number of different allowable formations for any given n.

2. **[25]** For this problem, our lawn is an $m \times n$ rectangular grid of clumps, that is, with m rows running east-west and n columns running north-south. To be even more explicit, we might say our clumps are at the lattice points

$$\{(x, y) \in \mathbb{Z}^2 \mid 0 \le x < n \text{ and } 0 \le y < m\}.$$

However, mobots are now allowed to be oriented to go either north or east only. So one allowable formation for $m = 2$, $n = 3$ might be as follows:

$$\begin{array}{ccc} \cdot & \rightarrow & \cdot \\ \uparrow & \rightarrow & \cdot \end{array}$$

Prove that the number of allowable formations for given m and n is $\dfrac{(m+n)!}{m!\,n!}$.

3. **[40]** In this problem, we stipulate that $m \ge n$, and the lawn is shaped differently. The clumps are now at the lattice points in a trapezoid:

$$\{(x, y) \in \mathbb{Z}^2 \mid 0 \le x < n \text{ and } 0 \le y < m + 1 - n + x\},$$

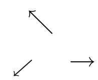
As in problem 2, mobots can be set to move either north or east. For given m and n, determine with proof the number of allowable formations.

4. **[15]** In this problem and the next, the lawn consists of points in a triangular grid of size n, so that for $n = 3$ the lawn looks like

$$\begin{array}{ccccc} & & \cdot & & \\ & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot \end{array}$$

Mobots are allowed to be oriented to the east, 30° west of north, or 30° west of south. Under these conditions, for any given n, what is the minimum number of mobots needed to now the lawn?

5. **[25]** With the same lawn and the same allowable mobot orientations as in the previous problem, let us call a formation "happy" if it is invariant under 120° rotations. (A rotation applies both to the positions of the mobots and to their orientations.) An example of a happy formation for $n = 2$ might be



Find the number of happy formations for a given n.

Polygons [110]

6. **[15]** Let n be an integer at least 5. At most how many diagonals of a regular n-gon can be simultaneously drawn so that no two are parallel? Prove your answer.

7. **[25]** Given a convex n-gon, $n \geq 4$, at most how many diagonals can be drawn such that each drawn diagonal intersects every other drawn diagonal either in the interior of the n-gon or at a vertex? Prove your answer.

8. **[15]** Given a regular n-gon with sides of length 1, what is the smallest radius r such that there is a non-empty intersection of n circles of radius r centered at the vertices of the n-gon? Give r as a formula in terms of n. Be sure to prove your answer.

9. **[40]** Let $n \geq 3$ be a positive integer. Prove that given any n angles $0 < \theta_1, \theta_2, \ldots, \theta_n < 180°$, such that their sum is $180(n-2)$ degrees, there exists a convex n-gon having exactly those angles, in that order.

10. **[15]** Suppose we have an n-gon such that each interior angle, measured in degrees, is a positive integer. Suppose further that all angles are less than 180°, and that all angles are different sizes. What is the maximum possible value of n? Prove your answer.

What do the following problems have in common? [170]

11. **[15]** The lottery cards of a certain lottery contain all nine-digit numbers that can be formed with the digits 1, 2 and 3. There is exactly one number on each lottery card. There are only red, yellow and blue lottery cards. Two lottery numbers that differ from each other in all nine digits always appear on cards of different color. Someone draws a red card and a yellow card. The red card has the number 122 222 222 and the yellow card has the number 222 222 222. The first prize goes to the lottery card with the number 123 123 123. What color(s) can it possibly have? Prove your answer.

12. **[25]** A $3 \times 3 \times 3$ cube is built from 27 unit cubes. Suddenly five of those cubes mysteriously teleport away. What is the minimum possible surface area of the remaining solid? Prove your answer.

13. **[40]** Having lost a game of checkers and my temper, I dash all the pieces to the ground but one. This last checker, which is perfectly circular in shape, remains completely on the board, and happens to cover equal areas of red and black squares. Prove that the center of this piece must lie on a boundary between two squares (or at a junction of four).

14. **[40]** A number n is called *bummed out* if there is exactly one ordered pair of positive integers (x, y) such that
$$\lfloor x^2/y \rfloor + \lfloor y^2/x \rfloor = n.$$
Find all bummed out numbers.

15. **[50]** Find, with proof, all positive integer palindromes whose square is also a palindrome.

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Team Round B

Mobotics [135]

Spring is finally here in Cambridge, and it's time to mow our lawn. For the purpose of these problems, our lawn consists of little *clumps* of grass arranged in an $m \times n$ rectangular grid, that is, with m rows running east-west and n columns running north-south. To be even more explicit, we might say our clumps are at the lattice points

$$\{(x, y) \in \mathbb{Z}^2 \mid 0 \le x < n \text{ and } 0 \le y < m\}.$$

Our machinery consists of a fleet of identical *mowbots* (or "mobots" for short). A mobot is a lawn-mowing machine. To mow our lawn, we begin by choosing a *formation*: we place as many mobots as we want at various clumps of grass and orient each mobot's head in a certain direction, either north or east (*not* south or west). At the blow of a whistle, each mobot starts moving in the direction we've chosen, mowing every clump of grass in its path (including the clump it starts on) until it goes off the lawn.

Because the spring is so young, our lawn is rather delicate. Consequently, we want to make sure that every clump of grass is mowed once and only once. We will not consider formations that do not meet this criterion.

One more thing: two formations are considered "different" if there exists a clump of grass for which either (1) for exactly one of the formations does a mobot starts on that clump, or (2) there are mobots starting on this clump for both the formations, but they're oriented in different directions.

As an example, one allowable formation for $m = 2$, $n = 3$ might be as follows:

$$\begin{array}{ccc} \cdot & \rightarrow & \cdot \\ \uparrow & \rightarrow & \cdot \end{array}$$

1. **[25]** Prove that the maximum number of mobots you need to mow your lawn is $m + n - 1$.

2. **[40]** Prove that the minimum number of mobots you need to mow your lawn is $\min\{m, n\}$.

3. **[15]** Prove that, given any formation, each mobot may be colored in one of three colors — say, white, black, and blue — such that no two adjacent clumps of grass are mowed by different mobots of the same color. Two clumps of grass are adjacent if the distance between them is 1. In your proof, you may use the Four-Color Theorem if you're familiar with it.

4. **[15]** For $n = m = 4$, find a formation with 6 mobots for which there are exactly 12 ways to color the mobots in three colors as in problem 3. (No proof is necessary.)

5. **[40]** For $n, m \ge 3$, prove that a formation has exactly six possible colorings satisfying the conditions in problem 3 if and only if there is a mobot that starts at $(1, 1)$.

Polygons [130]

6. **[15]** Suppose we have a regular hexagon and draw all its sides and diagonals. Into how many regions do the segments divide the hexagon? (No proof is necessary.)

7. **[25]** Suppose we have an octagon with all angles of $135°$, and consecutive sides of alternating length 1 and $\sqrt{2}$. We draw all its sides and diagonals. Into how many regions do the segments divide the octagon? (No proof is necessary.)

8. **[25]** A regular 12-sided polygon is inscribed in a circle of radius 1. How many chords of the circle that join two of the vertices of the 12-gon have lengths whose squares are rational? (No proof is necessary.)

9. **[25]** Show a way to construct an equiangular hexagon with side lengths 1, 2, 3, 4, 5, and 6 (not necessarily in that order).

10. **[40]** Given a convex n-gon, $n \ge 4$, at most how many diagonals can be drawn such that each drawn diagonal intersects every other drawn diagonal strictly in the interior of the n-gon? Prove that your answer is correct.

What do the following problems have in common? [135]

11. **[15]** Find the largest positive integer n such that $1! + 2! + 3! + \cdots + n!$ is a perfect square. Prove that your answer is correct.

12. **[15]** Find all ordered triples (x, y, z) of positive reals such that $x + y + z = 27$ and $x^2 + y^2 + z^2 - xy - yz - zx = 0$. Prove that your answer is correct.

13. **[25]** Four circles with radii $1, 2, 3$, and r are externally tangent to one another. Compute r. (No proof is necessary.)

14. **[40]** Find the prime factorization of

$$2006^2 \cdot 2262 - 669^2 \cdot 3599 + 1593^2 \cdot 1337.$$

(No proof is necessary.)

15. **[40]** Let a, b, c, d be real numbers so that c, d are not both 0. Define the function

$$m(x) = \frac{ax + b}{cx + d}$$

on all real numbers x except possibly $-d/c$, in the event that $c \neq 0$. Suppose that the equation $x = m(m(x))$ has at least one solution that is not a solution of $x = m(x)$. Find all possible values of $a + d$. Prove that your answer is correct.

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Individual Round: Algebra Test

1. [**3**] Compute
$$\left\lfloor \frac{2007! + 2004!}{2006! + 2005!} \right\rfloor.$$
(Note that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

2. [**3**] Two reals x and y are such that $x - y = 4$ and $x^3 - y^3 = 28$. Compute xy.

3. [**4**] Three real numbers x, y, and z are such that $(x+4)/2 = (y+9)/(z-3) = (x+5)/(z-5)$. Determine the value of x/y.

4. [**4**] Compute
$$\frac{2^3 - 1}{2^3 + 1} \cdot \frac{3^3 - 1}{3^3 + 1} \cdot \frac{4^3 - 1}{4^3 + 1} \cdot \frac{5^3 - 1}{5^3 + 1} \cdot \frac{6^3 - 1}{6^3 + 1}.$$

5. [**5**] A convex quadrilateral is determined by the points of intersection of the curves $x^4 + y^4 = 100$ and $xy = 4$; determine its area.

6. [**5**] Consider the polynomial $P(x) = x^3 + x^2 - x + 2$. Determine all real numbers r for which there exists a complex number z not in the reals such that $P(z) = r$.

7. [**5**] An infinite sequence of positive real numbers is defined by $a_0 = 1$ and $a_{n+2} = 6a_n - a_{n+1}$ for $n = 0, 1, 2, \cdots$ Find the possible value(s) of a_{2007}.

8. [**6**] Let $A := \mathbb{Q} \setminus \{0, 1\}$ denote the set of all rationals other than 0 and 1. A function $f : A \to \mathbb{R}$ has the property that for all $x \in A$,
$$f(x) + f\left(1 - \frac{1}{x}\right) = \log |x|.$$
Compute the value of $f(2007)$.

9. [**7**] The complex numbers $\alpha_1, \alpha_2, \alpha_3$, and α_4 are the four distinct roots of the equation $x^4 + 2x^3 + 2 = 0$. Determine the unordered set
$$\{\alpha_1\alpha_2 + \alpha_3\alpha_4, \alpha_1\alpha_3 + \alpha_2\alpha_4, \alpha_1\alpha_4 + \alpha_2\alpha_3\}.$$

10. [**8**] The polynomial $f(x) = x^{2007} + 17x^{2006} + 1$ has distinct zeroes $r_1, \ldots, r_{2007}$. A polynomial P of degree 2007 has the property that $P\left(r_j + \frac{1}{r_j}\right) = 0$ for $j = 1, \ldots, 2007$. Determine the value of $P(1)/P(-1)$.

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Individual Round: Geometry Test

1. [**3**] A cube of edge length $s > 0$ has the property that its surface area is equal to the sum of its volume and five times its edge length. Compute all possible values of s.

2. [**3**] A, B, C, and D are points on a circle, and segments $\overline{AC}$ and $\overline{BD}$ intersect at P, such that $AP = 8$, $PC = 1$, and $BD = 6$. Find BP, given that $BP < DP$.

3. [**4**] Circles ω_1, ω_2, and ω_3 are centered at M, N, and O, respectively. The points of tangency between ω_2 and ω_3, ω_3 and ω_1, and ω_1 and ω_2 are tangent at A, B, and C, respectively. Line MO intersects ω_3 and ω_1 again at P and Q respectively, and line AP intersects ω_2 again at R. Given that ABC is an equilateral triangle of side length 1, compute the area of PQR.

4. [**4**] Circle ω has radius 5 and is centered at O. Point A lies outside ω such that $OA = 13$. The two tangents to ω passing through A are drawn, and points B and C are chosen on them (one on each tangent), such that line BC is tangent to ω and ω lies outside triangle ABC. Compute $AB + AC$ given that $BC = 7$.

5. [**5**] Five marbles of various sizes are placed in a conical funnel. Each marble is in contact with the adjacent marble(s). Also, each marble is in contact all around the funnel wall. The smallest marble has a radius of 8, and the largest marble has a radius of 18. What is the radius of the middle marble?

6. [**5**] Triangle ABC has $\angle A = 90°$, side $BC = 25, AB > AC$, and area 150. Circle ω is inscribed in ABC, with M its point of tangency on AC. Line BM meets ω a second time at point L. Find the length of segment BL.

7. [**5**] Convex quadrilateral $ABCD$ has sides $AB = BC = 7$, $CD = 5$, and $AD = 3$. Given additionally that $m\angle ABC = 60°$, find BD.

8. [**6**] $ABCD$ is a convex quadrilateral such that $AB < AD$. The diagonal $\overline{AC}$ bisects $\angle BAD$, and $m\angle ABD = 130°$. Let E be a point on the interior of $\overline{AD}$, and $m\angle BAD = 40°$. Given that $BC = CD = DE$, determine $m\angle ACE$ in degrees.

9. [**7**] $\triangle ABC$ is right angled at A. D is a point on AB such that $CD = 1$. AE is the altitude from A to BC. If $BD = BE = 1$, what is the length of AD?

10. [**8**] $ABCD$ is a convex quadrilateral such that $AB = 2, BC = 3, CD = 7$, and $AD = 6$. It also has an incircle. Given that $\angle ABC$ is right, determine the radius of this incircle.

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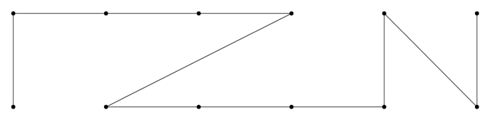
Individual Round: Combinatorics Test

1. [**3**] A committee of 5 is to be chosen from a group of 9 people. How many ways can it be chosen, if Biff and Jacob must serve together or not at all, and Alice and Jane refuse to serve with each other?

2. [**3**] How many 5-digit numbers $\overline{abcde}$ exist such that digits b and d are each the sum of the digits to their immediate left and right? (That is, $b = a + c$ and $d = c + e$.)

3. [**4**] Jack, Jill, and John play a game in which each randomly picks and then *replaces* a card from a standard 52 card deck, until a spades card is drawn. What is the probability that Jill draws the spade? (Jack, Jill, and John draw in that order, and the game repeats if no spade is drawn.)

4. [**4**] On the Cartesian grid, Johnny wants to travel from $(0, 0)$ to $(5, 1)$, and he wants to pass through all twelve points in the set $S = \{(i, j) \mid 0 \le i \le 1, 0 \le j \le 5, i, j \in \mathbb{Z}\}$. Each step, Johnny may go from one point in S to another point in S by a line segment connecting the two points. How many ways are there for Johnny to start at $(0, 0)$ and end at $(5, 1)$ so that he never crosses his own path?

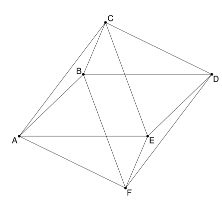


5. [**5**] Determine the number of ways to select a positive number of squares on an 8×8 chessboard such that no two lie in the same row or the same column and no chosen square lies to the left of and below another chosen square.

6. [**5**] Kevin has four red marbles and eight blue marbles. He arranges these twelve marbles randomly, in a ring. Determine the probability that no two red marbles are adjacent.

7. [**5**] Forty two cards are labeled with the natural numbers 1 through 42 and randomly shuffled into a stack. One by one, cards are taken off of the top of the stack until a card labeled with a prime number is removed. How many cards are removed on average?

8. [**6**] A set of six edges of a regular octahedron is called *Hamiltonian cycle* if the edges in some order constitute a single continuous loop that visits each vertex exactly once. How many ways are there to partition the twelve edges into two Hamiltonian cycles?



9. [**7**] Let S denote the set of all triples (i, j, k) of positive integers where $i + j + k = 17$. Compute

$$\sum_{(i,j,k) \in S} ijk.$$

10. [**8**] A subset S of the nonnegative integers is called *supported* if it contains 0, and $k + 8, k + 9 \in S$ for all $k \in S$. How many supported sets are there?

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Individual Round: Calculus Test

1. [**3**] Compute:

$$\lim_{x \to 0} \quad \frac{x^2}{1 - \cos(x)}$$

2. [**3**] Determine the real number a having the property that $f(a) = a$ is a relative minimum of $f(x) = x^4 - x^3 - x^2 + ax + 1$.

3. [**4**] Let a be a positive real number. Find the value of a such that the definite integral

$$\int_a^{a^2} \frac{\mathrm{d}x}{x + \sqrt{x}}$$

achieves its smallest possible value.

4. [**4**] Find the real number α such that the curve $f(x) = e^x$ is tangent to the curve $g(x) = \alpha x^2$.

5. [**5**] The function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x^2)f''(x) = f'(x)f'(x^2)$ for all real x. Given that $f(1) = 1$ and $f'''(1) = 8$, determine $f'(1) + f''(1)$.

6. [**5**] The elliptic curve $y^2 = x^3 + 1$ is tangent to a circle centered at $(4, 0)$ at the point (x_0, y_0). Determine the sum of all possible values of x_0.

7. [**5**] Compute

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot (n+1) \cdot (n+1)!}.$$

8. [**6**] Suppose that ω is a primitive 2007^{th} root of unity. Find $\left(2^{2007} - 1\right) \sum_{j=1}^{2006} \frac{1}{2 - \omega^j}$.

For this problem only, you may express your answer in the form $m \cdot n^k + p$, where m, n, k, and p are positive integers. Note that a number z is a *primitive* n^{th} *root of unity* if $z^n = 1$ and n is the smallest number amongst $k = 1, 2, \ldots, n$ such that $z^k = 1$.

9. [**7**] g is a twice differentiable function over the positive reals such that

$$g(x) + 2x^3 g'(x) + x^4 g''(x) = 0 \quad \text{for all positive reals } x. \qquad (1)$$

$$\lim_{x \to \infty} x g(x) = 1 \qquad (2)$$

Find the real number $\alpha > 1$ such that $g(\alpha) = 1/2$.

10. [**8**] Compute

$$\int_0^{\infty} \frac{e^{-x} \sin(x)}{x} dx$$

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Individual Round: General Test, Part 1

1. [**2**] Michael has 16 white socks, 3 blue socks, and 6 red socks in a drawer. Ever the lazy college student, he has overslept and is late for his favorite team's season-opener. Because he is now in such a rush to get from Harvard to Foxborough, he randomly takes socks from the drawer (one at a time) until he has a pair of the same color. What is the largest number of socks he could possibly withdraw in this fashion?

2. [**2**] Rectangle $ABCD$ has side lengths $AB = 12$ and $BC = 5$. Let P and Q denote the midpoints of segments AB and DP, respectively. Determine the area of triangle CDQ.

3. [**3**] A, B, C, and D are points on a circle, and segments $\overline{AC}$ and $\overline{BD}$ intersect at P, such that $AP = 8$, $PC = 1$, and $BD = 6$. Find BP, given that $BP < DP$.

4. [**3**] Let a and b be integer solutions to $17a + 6b = 13$. What is the smallest possible positive value for $a - b$?

5. [**4**] Find the smallest positive integer that is twice a perfect square and three times a perfect cube.

6. [**4**] The positive integer n is such that the numbers 2^n and 5^n start with the same digit when written in decimal notation; determine this common leading digit.

7. [**4**] Jack, Jill, and John play a game in which each randomly picks and then replaces a card from a standard 52 card deck, until a spades card is drawn. What is the probability that Jill draws the spade? (Jack, Jill, and John draw in that order, and the game repeats if no spade is drawn.)

8. [**5**] Determine the largest positive integer n such that there exist positive integers x, y, z so that

$$n^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx + 3x + 3y + 3z - 6$$

9. [**6**] I have four distinct rings that I want to wear on my right hand hand (five distinct fingers.) One of these rings is a Canadian ring that must be worn on a finger by itself, the rest I can arrange however I want. If I have two or more rings on the same finger, then I consider different orders of rings along the same finger to be different arrangements. How many different ways can I wear the rings on my fingers?

10. [**7**] $\alpha_1, \alpha_2, \alpha_3$, and α_4 are the complex roots of the equation $x^4 + 2x^3 + 2 = 0$. Determine the unordered set

$$\{\alpha_1\alpha_2 + \alpha_3\alpha_4, \alpha_1\alpha_3 + \alpha_2\alpha_4, \alpha_1\alpha_4 + \alpha_2\alpha_3\}.$$

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Individual Round: General Test, Part 2

1. [**2**] A cube of edge length $s > 0$ has the property that its surface area is equal to the sum of its volume and five times its edge length. Compute all possible values of s.

2. [**2**] A parallelogram has 3 of its vertices at (1,2), (3,8), and (4,1). Compute the sum of all possible x coordinates of the 4th vertex.

3. [**3**] Compute
$$\left\lfloor \frac{2007! + 2004!}{2006! + 2005!} \right\rfloor.$$
(Note that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

4. [**3**] Three brothers Abel, Banach, and Gauss each have portable music players that can share music with each other. Initially, Abel has 9 songs, Banach has 6 songs, and Gauss has 3 songs, and none of these songs are the same. One day, Abel flips a coin to randomly choose one of his brothers and he adds all of that brother's songs to his collection. The next day, Banach flips a coin to randomly choose one of his brothers and he adds all of that brother's collection of songs to his collection. Finally, each brother randomly plays a song from his collection with each song in his collection being equally likely to be chosen. What is the probability that they all play the same song?

5. [**4**] A best of 9 series is to be played between two teams. That is, the first team to win 5 games is the winner. One of the teams, the Mathletes, has a 2/3 chance of winning any given game. What is the probability that the winner is determined in the 7th game?

6. [**4**] Circle ω has radius 5 and is centered at O. Point A lies outside ω such that $OA = 13$. The two tangents to ω passing through A are drawn, and points B and C are chosen on them (one on each tangent), such that line BC is tangent to ω and ω lies outside triangle ABC. Compute $AB + AC$ given that $BC = 7$.

7. [**4**] My friend and I are playing a game with the following rules: If one of us says an integer n, the opponent then says an integer of their choice between $2n$ and $3n$, inclusive. Whoever first says 2007 or greater loses the game, and their opponent wins. I must begin the game by saying a positive integer less than 10. With how many of them can I guarantee a win?

8. [**5**] Compute the number of sequences of numbers $a_1, a_2, \ldots, a_{10}$ such that

   I. $a_i = 0$ or 1 for all i
   II. $a_i \cdot a_{i+1} = 0$ for $i = 1, 2, \ldots, 9$
   III. $a_i \cdot a_{i+2} = 0$ for $i = 1, 2, \ldots, 8$.

9. [**6**] Let $A := \mathbb{Q} \setminus \{0, 1\}$ denote the set of all rationals other than 0 and 1. A function $f : A \to \mathbb{R}$ has the property that for all $x \in A$,
$$f(x) + f\left(1 - \frac{1}{x}\right) = \log |x|.$$
Compute the value of $f(2007)$.

10. [**7**] $ABCD$ is a convex quadrilateral such that $AB = 2, BC = 3, CD = 7$, and $AD = 6$. It also has an incircle. Given that $\angle ABC$ is right, determine the radius of this incircle.

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Guts Round

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Note that there are just 36 problems in the Guts round this year.

1. [**5**] Define the sequence of positive integers a_n recursively by $a_1 = 7$ and $a_n = 7^{a_{n-1}}$ for all $n \geq 2$. Determine the last two digits of a_{2007}.

2. [**5**] A candy company makes 5 colors of jellybeans, which come in equal proportions. If I grab a random sample of 5 jellybeans, what is the probability that I get exactly 2 distinct colors?

3. [**5**] The equation $x^2 + 2x = i$ has two complex solutions. Determine the product of their real parts.

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4. [**6**] A sequence consists of the digits 122333444455555... such that the each positive integer n is repeated n times, in increasing order. Find the sum of the 4501st and 4052nd digits of this sequence.

5. [**6**] Compute the largest positive integer such that $\frac{2007!}{2007^n}$ is an integer.

6. [**6**] There are three video game systems: the Paystation, the WHAT, and the ZBoz2π, and none of these systems will play games for the other systems. Uncle Riemann has three nephews: Bernoulli, Galois, and Dirac. Bernoulli owns a Paystation and a WHAT, Galois owns a WHAT and a ZBoz2π, and Dirac owns a ZBoz2π and a Paystation. A store sells 4 different games for the Paystation, 6 different games for the WHAT, and 10 different games for the ZBoz2π. Uncle Riemann does not understand the difference between the systems, so he walks into the store and buys 3 random games (not necessarily distinct) and randomly hands them to his nephews. What is the probability that each nephew receives a game he can play?

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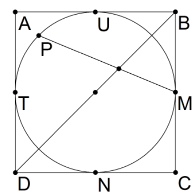
7. [**7**] A student at Harvard named Kevin
   Was counting his stones by 11
   He messed up n times
   And instead counted 9s
   And wound up at 2007.

   How many values of n could make this limerick true?

8. [**7**] A circle inscribed in a square,
   Has two chords as shown in a pair.
   It has radius 2,
   And P bisects TU.
   The chords' intersection is where?



   Answer the question by giving the distance of the point of intersection from the center of the circle.

9. [**7**] I ponder some numbers in bed,
   All products of three primes I've said,
   Apply ϕ they're still fun:
   now Elev'n cubed plus one.
   What numbers could be in my head?

   $n = 37^2 \cdot 3 \ldots$
   $\phi(n) =$
   $11^3 + 1$?

. . . .

10. [**8**] Let A_{12} denote the answer to problem 12. There exists a unique triple of digits (B, C, D) such that $10 > A_{12} > B > C > D > 0$ and

$$\overline{A_{12}BCD} - \overline{DCBA_{12}} = \overline{BDA_{12}C},$$

   where $\overline{A_{12}BCD}$ denotes the four digit base 10 integer. Compute $B + C + D$.

11. [**8**] Let A_{10} denote the answer to problem 10. Two circles lie in the plane; denote the lengths of the internal and external tangents between these two circles by x and y, respectively. Given that the product of the radii of these two circles is 15/2, and that the distance between their centers is A_{10}, determine $y^2 - x^2$.

12. [**8**] Let A_{11} denote the answer to problem 11. Determine the smallest prime p such that the arithmetic sequence $p, p + A_{11}, p + 2A_{11}, \ldots$ begins with the largest possible number of primes.

    There is just one triple of possible (A_{10}, A_{11}, A_{12}) of answers to these three problems. Your team will receive credit only for answers matching these. (So, for example, submitting a wrong answer for problem 11 will not alter the correctness of your answer to problem 12.)

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13. [**9**] Determine the largest integer n such that $7^{2048} - 1$ is divisible by 2^n.

14. [**9**] We are given some similar triangles. Their areas are $1^2, 3^2, 5^2 \ldots$, and 49^2. If the smallest triangle has a perimeter of 4, what is the sum of all the triangles' perimeters?

15. [**9**] Points A, B, and C lie in that order on line ℓ, such that $AB = 3$ and $BC = 2$. Point H is such that CH is perpendicular to ℓ. Determine the length CH such that $\angle AHB$ is as large as possible.

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16. [**10**] Let ABC be a triangle with $AB = 7, BC = 9$, and $CA = 4$. Let D be the point such that $AB \parallel CD$ and $CA \parallel BD$. Let R be a point within triangle BCD. Lines ℓ and m going through R are parallel to CA and AB respectively. Line ℓ meets AB and BC at P and P' respectively, and m meets CA and BC at Q and Q' respectively. If S denotes the largest possible sum of the areas of triangles $BPP', RP'Q'$, and CQQ', determine the value of S^2.

17. [**10**] During the regular season, Washington Redskins achieve a record of 10 wins and 6 losses. Compute the probability that their wins came in three streaks of consecutive wins, assuming that all possible arrangements of wins and losses are equally likely. (For example, the record LLWWWWWLWWL-WWWLL contains three winning streaks, while WWWWWWWLLLLLLWWW has just two.)

18. [**10**] Convex quadrilateral $ABCD$ has right angles $\angle A$ and $\angle C$ and is such that $AB = BC$ and $AD = CD$. The diagonals AC and BD intersect at point M. Points P and Q lie on the circumcircle of triangle AMB and segment CD, respectively, such that points P, M, and Q are collinear. Suppose that $m\angle ABC = 160°$ and $m\angle QMC = 40°$. Find $MP \cdot MQ$, given that $MC = 6$.

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19. [**10**] Define $x \star y = \frac{\sqrt{x^2+3xy+y^2-2x-2y+4}}{xy+4}$. Compute

$$((\cdots((2007 \star 2006) \star 2005) \star \cdots) \star 1).$$

20. [**10**] For a a positive real number, let x_1, x_2, x_3 be the roots of the equation $x^3 - ax^2 + ax - a = 0$. Determine the smallest possible value of $x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3$.

21. [**10**] Bob the bomb-defuser has stumbled upon an active bomb. He opens it up, and finds the red and green wires conveniently located for him to cut. Being a seasoned member of the bomb-squad, Bob quickly determines that it is the green wire that he should cut, and puts his wirecutters on the green wire. But just before he starts to cut, the bomb starts to count down, ticking every second. Each time the bomb ticks, starting at time $t = 15$ seconds, Bob panics and has a certain chance to move his wirecutters to the other wire. However, he is a rational man even when panicking, and has a $\frac{1}{2t^2}$ chance of switching wires at time t, regardless of which wire he is about to cut. When the bomb ticks at $t = 1$, Bob cuts whatever wire his wirecutters are on, without switching wires. What is the probability that Bob cuts the green wire?

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22. [**12**] The sequence $\{a_n\}_{n\geq 1}$ is defined by $a_{n+2} = 7a_{n+1} - a_n$ for positive integers n with initial values $a_1 = 1$ and $a_2 = 8$. Another sequence, $\{b_n\}$, is defined by the rule $b_{n+2} = 3b_{n+1} - b_n$ for positive integers n together with the values $b_1 = 1$ and $b_2 = 2$. Find $\gcd(a_{5000}, b_{501})$.

23. [**12**] In triangle ABC, $\angle ABC$ is obtuse. Point D lies on side AC such that $\angle ABD$ is right, and point E lies on side AC between A and D such that BD bisects $\angle EBC$. Find CE, given that $AC = 35, BC = 7$, and $BE = 5$.

24. [**12**] Let x, y, n be positive integers with $n > 1$. How many ordered triples (x, y, n) of solutions are there to the equation $x^n - y^n = 2^{100}$?

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25. [**12**] Two real numbers x and y are such that $8y^4 + 4x^2y^2 + 4xy^2 + 2x^3 + 2y^2 + 2x = x^2 + 1$. Find all possible values of $x + 2y^2$.

26. [**12**] $ABCD$ is a cyclic quadrilateral in which $AB = 4, BC = 3, CD = 2$, and $AD = 5$. Diagonals AC and BD intersect at X. A circle ω passes through A and is tangent to BD at X. ω intersects AB and AD at Y and Z respectively. Compute YZ/BD.

27. [**12**] Find the number of 7-tuples $(n_1, \ldots, n_7)$ of integers such that

$$\sum_{i=1}^{7} n_i^6 = 96957.$$

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28. [**15**] Compute the circumradius of cyclic hexagon $ABCDEF$, which has side lengths $AB = BC = 2, CD = DE = 9$, and $EF = FA = 12$.

29. [**15**] A sequence $\{a_n\}_{n\geq 1}$ of positive reals is defined by the rule $a_{n+1}a_{n-1}^5 = a_n^4 a_{n-2}^2$ for integers $n > 2$ together with the initial values $a_1 = 8$ and $a_2 = 64$ and $a_3 = 1024$. Compute

$$\sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + \cdots}}}$$

30. [**15**] $ABCD$ is a cyclic quadrilateral in which $AB = 3, BC = 5, CD = 6$, and $AD = 10$. M, I, and T are the feet of the perpendiculars from D to lines AB, AC, and BC respectively. Determine the value of MI/IT.

. . . .

31. [**18**] A sequence $\{a_n\}_{n\geq 0}$ of real numbers satisfies the recursion $a_{n+1} = a_n^3 - 3a_n^2 + 3$ for all positive integers n. For how many values of a_0 does $a_{2007} = a_0$?

32. [**18**] Triangle ABC has $AB = 4, BC = 6$, and $AC = 5$. Let O denote the circumcenter of ABC. The circle Γ is tangent to and surrounds the circumcircles of triangles AOB, BOC, and AOC. Determine the diameter of Γ.

33. [**18**] Compute

$$\int_1^2 \frac{9x+4}{x^5+3x^2+x}dx.$$

(No, your TI-89 doesn't know how to do this one. Yes, the end is near.)

. . . .

10th HARVARD-MIT MATHEMATICS TOURNAMENT, 24 FEBRUARY 2007 — GUTS ROUND

34. [?] *The Game.* Eric and Greg are watching their new favorite TV show, *The Price is Right.* Bob Barker recently raised the intellectual level of his program, and he begins the latest installment with bidding on following question: How many Carmichael numbers are there less than 100,000?

Each team is to list one nonnegative integer not greater than 100,000. Let X denote the answer to Bob's question. The teams listing N, a maximal bid (of those submitted) not greater than X, will receive N points, and all other teams will neither receive nor lose points. (A Carmichael number is an odd composite integer n such that n divides $a^{n-1} - 1$ for all integers a relatively prime to n with $1 < a < n$.)

35. [$\leq$ **25**] *The Algorithm.* There are thirteen broken computers situated at the following set S of thirteen points in the plane:

$$\begin{array}{ccc} A = (1, 10) & B = (976, 9) & C = (666, 87) \\ D = (377, 422) & E = (535, 488) & F = (775, 488) \\ G = (941, 500) & H = (225, 583) & I = (388, 696) \\ J = (3, 713) & K = (504, 872) & L = (560, 934) \\ & M = (22, 997) & \end{array}$$

At time $t = 0$, a repairman begins moving from one computer to the next, traveling continuously in straight lines at unit speed. Assuming the repairman begins and A and fixes computers instantly, what path does he take to minimize the *total downtime* of the computers? List the points he visits in order. Your score will be $\lfloor \frac{N}{40} \rfloor$, where

$$N = 1000 + \lfloor \text{the optimal downtime} \rfloor - \lfloor \text{your downtime} \rfloor,$$

or 0, whichever is greater. By total downtime we mean the sum

$$\sum_{P \in S} t_P,$$

where t_P is the time at which the repairman reaches P.

36. [**25**] *The Marathon.* Let ω denote the incircle of triangle ABC. The segments BC, CA, and AB are tangent to ω at D, E, and F, respectively. Point P lies on EF such that segment PD is perpendicular to BC. The line AP intersects BC at Q. The circles ω_1 and ω_2 pass through B and C, respectively, and are tangent to AQ at Q; the former meets AB again at X, and the latter meets AC again at Y. The line XY intersects BC at Z. Given that $AB = 15, BC = 14$, and $CA = 13$, find $\lfloor XZ \cdot YZ \rfloor$.

10^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 24 February 2007

Team Round: A Division

Σ, τ, and You: Fun at Fraternities? [270]

A *number theoretic function* is a function whose domain is the set of positive integers. A *multiplicative* number theoretic function is a number theoretic function f such that $f(mn) = f(m)f(n)$ for all pairs of relatively prime positive integers m and n. Examples of multiplicative number theoretic functions include σ, τ, ϕ, and μ, defined as follows. For each positive integer n,

- The *sum-of-divisors function*, $\sigma(n)$, is the sum of all positive integer divisors of n. If $p_1, \ldots, p_i$ are distinct primes and $e_1, \ldots, e_i$ are positive integers,

$$\sigma\left(p_1^{e_1} \cdots p_i^{e_i}\right) = \prod_{k=1}^{i} \left(1 + p_k + \cdots + p_k^{e_k}\right) = \prod_{k=1}^{i} \frac{p_k^{e_k+1} - 1}{p_k - 1}.$$

- The *divisor function*, $\tau(n)$, is the number of positive integer divisors of n. It can be computed by the formula

$$\tau\left(p_1^{e_1} \cdots p_i^{e_i}\right) = (e_1 + 1) \cdots (e_i + 1),$$

  where $p_1, \ldots, p_i$ and $e_1, \ldots, e_i$ are as above.

- Euler's *totient* function, $\phi(n)$, is the number of positive integers $k \leq n$ such that k and n are relatively prime. For $p_1, \ldots, p_i$ and $e_1, \ldots, e_i$ as above, the phi function satisfies

$$\phi\left(p_1^{e_1} \cdots p_i^{e_i}\right) = \prod_{k=1}^{i} p_k^{e_k - 1}(p_k - 1).$$

- The *Möbius function*, $\mu(n)$, is equal to either 1, -1, or 0. An integer is called *square-free* if it is not divisible by the square of any prime. If n is a square-free positive integer having an even number of distinct prime divisors, $\mu(n) = 1$. If n is a square-free positive integer having an odd number of distinct prime divisors, $\mu(n) = -1$. Otherwise, $\mu(n) = 0$.

The Möbius function has a number of peculiar properties. For example, if f and g are number theoretic functions such that

$$g(n) = \sum_{d|n} f(d),$$

for all positive integers n, then

$$f(n) = \sum_{d|n} g(d)\mu\left(\frac{n}{d}\right).$$

This is known as *Möbius inversion.* In proving the following problems, *you may use any of the preceding assertions without proving them. You may also cite the results of previous problems, even if you were unable to prove them.*

1. [**15**] Evaluate the functions $\phi(n), \sigma(n)$, and $\tau(n)$ for $n = 12, n = 2007$, and $n = 2^{2007}$.

2. [**20**] Solve for the positive integer(s) n such that $\phi(n^2) = 1000\phi(n)$.

3. [**25**] Prove that for every integer n greater than 1,

$$\sigma(n)\phi(n) \leq n^2 - 1.$$

   When does equality hold?

4. [**25**] Let F and G be two multiplicative functions, and define for positive integers n,

$$H(n) = \sum_{d|n} F(d)G\left(\frac{n}{d}\right).$$

The number theoretic function H is called the *convolution* of F and G. Prove that H is multiplicative.

5. [**30**] Prove the identity

$$\sum_{d|n} \tau(d)^3 = \left(\sum_{d|n} \tau(d)\right)^2.$$

6. [**25**] Show that for positive integers n,

$$\sum_{d|n} \phi(d) = n.$$

7. [**25**] Show that for positive integers n,

$$\sum_{d|n} \frac{\mu(d)}{d} = \frac{\phi(n)}{n}.$$

8. [**30**] Determine with proof, a simple closed form expression for

$$\sum_{d|n} \phi(d)\tau\left(\frac{n}{d}\right).$$

9. [**35**] Find all positive integers n such that

$$\sum_{k=1}^{n} \phi(k) = \frac{3n^2+5}{8}.$$

10. [**40**] Find all pairs (n, k) of positive integers such that

$$\sigma(n)\phi(n) = \frac{n^2}{k}.$$

Grab Bag - Miscellaneous Problems [130]

11. [**30**] Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$\begin{aligned} f(x)f(y) &= f(x) + f(y) - f(xy) \\ 1 + f(x+y) &= f(xy) + f(x)f(y) \end{aligned}$$

for all rational numbers x, y.

12. [**30**] Let $ABCD$ be a cyclic quadrilateral, and let P be the intersection of its two diagonals. Points R, S, T, and U are feet of the perpendiculars from P to sides AB, BC, CD, and AD, respectively. Show that quadrilateral $RSTU$ is bicentric if and only if $AC \perp BD$. (Note that a quadrilateral is called *inscriptible* if it has an incircle; a quadrilateral is called *bicentric* if it is both cyclic and inscriptible.)

13. [**30**] Find all nonconstant polynomials $P(x)$, with real coefficients and having only real zeros, such that $P(x+1)P(x^2-x+1) = P(x^3+1)$ for all real numbers x.

14. [**40**] Find an explicit, closed form formula for

$$\sum_{k=1}^{n} \frac{k \cdot (-1)^k \cdot \binom{n}{k}}{n+k+1}.$$

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Team Round: B Division

Compute $(x-a)(x-b)\cdots(x-z)$ - Short Answer [200]

For this section, your team should give only the anwers to the problems.

1. [**20**] Find the sum of the positive integer divisors of 2^{2007}.

2. [**20**] The four sides of quadrilateral $ABCD$ are equal in length. Determine the perimeter of $ABCD$ given that it has area 120 and $AC = 10$.

3. [**20**] Five people are crowding into a booth against a wall at a noisy restaurant. If at most three can fit on one side, how many seating arrangements accomodate them all?

4. [**20**] Thomas and Michael are just two people in a large pool of well qualified candidates for appointment to a problem writing committee for a prestigious college math contest. It is 40 times more likely that both will serve if the size of the committee is increased from its traditional 3 members to a whopping n members. Determine n. (Each person in the pool is equally likely to be chosen.)

5. [**20**] The curves $y = x^2(x-3)^2$ and $y = (x^2-1)(x-2)$ intersect at a number of points in the real plane. Determine the sum of the x-coordinates of these points of intersection.

6. [**20**] Andrew has a fair six sided die labeled with 1 through 6 as usual. He tosses it repeatedly, and on every third roll writes down the number facing up as long as it is not the 6. He stops as soon as the last two numbers he has written down are squares or one is a prime and the other is a square. What is the probability that he stops after writing squares consecutively?

7. [**20**] Three positive reals x, y, and z are such that

$$\begin{aligned} x^2 + 2(y-1)(z-1) &= 85 \\ y^2 + 2(z-1)(x-1) &= 84 \\ z^2 + 2(x-1)(y-1) &= 89. \end{aligned}$$

   Compute $x + y + z$.

8. [**20**] Find the *positive* real number(s) x such that $\frac{1}{2}\left(3x^2-1\right) = \left(x^2-50x-10\right)\left(x^2+25x+5\right)$.

9. [**20**] Cyclic quadrilateral $ABCD$ has side lengths $AB = 1, BC = 2, CD = 3$, and $AD = 4$. Determine AC/BD.

10. [**20**] A positive real number x is such that

$$\sqrt[3]{1-x^3} + \sqrt[3]{1+x^3} = 1.$$

   Find x^2.

Adult Acorns - Gee, I'm a Tree! [200]

In this section of the team round, your team will derive some basic results concerning *tangential* quadrilaterals. Tangential quadrilaterals have an *incircle*, or a circle lying within them that is tangent to all four sides. If a quadrilateral has an incircle, then the center of this circle is the *incenter* of the quadrilateral. As you shall see, tangential quadrilaterals are related to cyclic quadrilaterals. For reference, a review of cyclic quadrilaterals is given at the end of this section.

Your answers for this section of the team test should be proofs. *Note that you may use any standard facts about cyclic quadrilaterals, such as those listed at the end of this test, without proving them. Additionally, you may cite the results of previous problems, even if you were unable to prove them.*

For these problems, $ABCD$ is a tangential quadrilateral having incenter I. For the first three problems, the point P is constructed such that triangle PAB is similar to triangle IDC and lies outside $ABCD$.

1. **[30]** Show that $PAIB$ is cyclic by proving that $\angle IAP$ is supplementary to $\angle PBI$.

2. **[40]** Show that triangle PAI is similar to triangle BIC. Then conclude that

$$PA = \frac{PI}{BC} \cdot BI.$$

3. **[25]** Deduce from the above that

$$\frac{BC}{AD} \cdot \frac{AI}{BI} \cdot \frac{DI}{CI} = 1.$$

4. **[25]** Show that $AB + CD = AD + BC$. Use the above to conclude that for some positive number α,

$$AB = \alpha \cdot \left(\frac{AI}{CI} + \frac{BI}{DI} \right) \qquad BC = \alpha \cdot \left(\frac{BI}{DI} + \frac{CI}{AI} \right)$$
$$CD = \alpha \cdot \left(\frac{CI}{AI} + \frac{DI}{BI} \right) \qquad DA = \alpha \cdot \left(\frac{DI}{BI} + \frac{AI}{CI} \right).$$

5. **[40]** Show that

$$AB \cdot BC = BI^2 + \frac{AI \cdot BI \cdot CI}{DI}.$$

6. **[40]** Let the incircle of $ABCD$ be tangent to sides AB, BC, CD, and AD at points P, Q, R, and S, respectively. Show that $ABCD$ is cyclic if and only if $PR \perp QS$.

A brief review of cyclic Quadrilaterals.

The following discussion of cyclic quadrilaterals is included for reference. Any of the results given here may be cited without proof in your writeups.

A *cyclic quadrilateral* is a quadrilateral whose four vertices lie on a circle called the *circumcircle* (the circle is unique if it exists.) If a quadrilateral has a circumcircle, then the center of this circumcircle is called the *circumcenter* of the quadrilateral. For a convex quadrilateral $ABCD$, the following are equivalent:

- Quadrilateral $ABCD$ is cyclic;
- $\angle ABD = \angle ACD$ (or $\angle BCA = \angle BDA$, etc.);
- Angles $\angle ABC$ and $\angle CDA$ are *supplementary*, that is, $m\angle ABC + m\angle CDA = 180°$ (or angles $\angle BCD$ and $\angle BAD$ are supplementary);

Cyclic quadrilaterals have a number of interesting properties. A cyclic quadrilateral $ABCD$ satisfies

$$AC \cdot BD = AB \cdot CD + AD \cdot BC,$$

a result known as *Ptolemy's theorem.* Another result, typically called *Power of a Point*, asserts that given a circle ω, a point P anywhere in the plane of ω, and a line ℓ through P intersecting ω at points A and B, the value of $AP \cdot BP$ is independent of ℓ; i.e., if a second line ℓ' through P intersects ω at A' and B', then $AP \cdot BP = A'P \cdot B'P$. This second theorem is proved via similar triangles. Say P lies outside of ω, that ℓ and ℓ' are as before and that A and A' lie on segments BP and $B'P$ respectively. Then triangle $AA'P$ is similar to triangle $B'BP$ because the triangles share an angle at P and we have

$$m\angle AA'P = 180° - m\angle B'A'A = m\angle ABB' = m\angle PBB'.$$

The case where $A = B$ is valid and describes the tangents to ω. A similar proof works for P inside ω.

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Individual Round: Algebra Test

1. [**3**] Positive real numbers x, y satisfy the equations $x^2 + y^2 = 1$ and $x^4 + y^4 = \frac{17}{18}$. Find xy.

2. [**3**] Let $f(n)$ be the number of times you have to hit the $\sqrt{\ }$ key on a calculator to get a number less than 2 starting from n. For instance, $f(2) = 1, f(5) = 2$. For how many $1 < m < 2008$ is $f(m)$ odd?

3. [**4**] Determine all real numbers a such that the inequality $|x^2 + 2ax + 3a| \leq 2$ has exactly one solution in x.

4. [**4**] The function f satisfies

$$f(x) + f(2x + y) + 5xy = f(3x - y) + 2x^2 + 1$$

for all real numbers x, y. Determine the value of $f(10)$.

5. [**5**] Let $f(x) = x^3 + x + 1$. Suppose g is a cubic polynomial such that $g(0) = -1$, and the roots of g are the squares of the roots of f. Find $g(9)$.

6. [**5**] A *root of unity* is a complex number that is a solution to $z^n = 1$ for some positive integer n. Determine the number of roots of unity that are also roots of $z^2 + az + b = 0$ for some integers a and b.

7. [**5**] Compute $\displaystyle\sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \frac{k}{2^{n+k}}$.

8. [**6**] Compute $\arctan(\tan 65^\circ - 2 \tan 40^\circ)$. (Express your answer in degrees.)

9. [**7**] Let S be the set of points (a, b) with $0 \leq a, b \leq 1$ such that the equation

$$x^4 + ax^3 - bx^2 + ax + 1 = 0$$

has at least one real root. Determine the area of the graph of S.

10. [**8**] Evaluate the infinite sum

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{5^n}.$$

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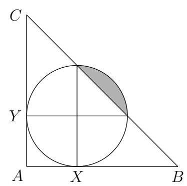
Individual Round: Geometry Test

1. [**3**] How many different values can $\angle ABC$ take, where A,B,C are distinct vertices of a cube?

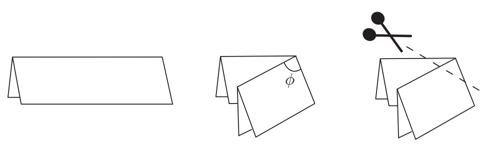
2. [**3**] Let ABC be an equilateral triangle. Let Ω be its incircle (circle inscribed in the triangle) and let ω be a circle tangent externally to Ω as well as to sides AB and AC. Determine the ratio of the radius of Ω to the radius of ω.

3. [**4**] Let ABC be a triangle with $\angle BAC = 90°$. A circle is tangent to the sides AB and AC at X and Y respectively, such that the points on the circle diametrically opposite X and Y both lie on the side BC. Given that $AB = 6$, find the area of the portion of the circle that lies outside the triangle.



4. [**4**] In a triangle ABC, take point D on BC such that $DB = 14, DA = 13, DC = 4$, and the circumcircle of ADB is congruent to the circumcircle of ADC. What is the area of triangle ABC?

5. [**5**] A piece of paper is folded in half. A second fold is made at an angle ϕ ($0° < \phi < 90°$) to the first, and a cut is made as shown below.



When the piece of paper is unfolded, the resulting hole is a polygon. Let O be one of its vertices. Suppose that all the other vertices of the hole lie on a circle centered at O, and also that $\angle XOY = 144°$, where X and Y are the the vertices of the hole adjacent to O. Find the value(s) of ϕ (in degrees).

6. [**5**] Let ABC be a triangle with $\angle A = 45°$. Let P be a point on side BC with $PB = 3$ and $PC = 5$. Let O be the circumcenter of ABC. Determine the length OP.

7. [**6**] Let C_1 and C_2 be externally tangent circles with radius 2 and 3, respectively. Let C_3 be a circle internally tangent to both C_1 and C_2 at points A and B, respectively. The tangents to C_3 at A and B meet at T, and $TA = 4$. Determine the radius of C_3.

8. [**6**] Let ABC be an equilateral triangle with side length 2, and let Γ be a circle with radius $\frac{1}{2}$ centered at the center of the equilateral triangle. Determine the length of the shortest path that starts somewhere on Γ, visits all three sides of ABC, and ends somewhere on Γ (not necessarily at the starting point). Express your answer in the form of $\sqrt{p} - q$, where p and q are rational numbers written as reduced fractions.

9. [**7**] Let ABC be a triangle, and I its incenter. Let the incircle of ABC touch side BC at D, and let lines BI and CI meet the circle with diameter AI at points P and Q, respectively. Given $BI = 6, CI = 5, DI = 3$, determine the value of $(DP/DQ)^2$.

10. [**7**] Let ABC be a triangle with $BC = 2007$, $CA = 2008$, $AB = 2009$. Let ω be an excircle of ABC that touches the line segment BC at D, and touches extensions of lines AC and AB at E and F, respectively (so that C lies on segment AE and B lies on segment AF). Let O be the center of ω. Let ℓ be the line through O perpendicular to AD. Let ℓ meet line EF at G. Compute the length DG.

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Individual Round: Combinatorics Test

1. [**3**] A $3 \times 3 \times 3$ cube composed of 27 unit cubes rests on a horizontal plane. Determine the number of ways of selecting two distinct unit cubes from a $3 \times 3 \times 1$ block (the order is irrelevant) with the property that the line joining the centers of the two cubes makes a 45° angle with the horizontal plane.

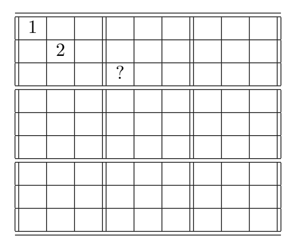
2. [**3**] Let $S = \{1, 2, \ldots, 2008\}$. For any nonempty subset $A \subset S$, define $m(A)$ to be the median of A (when A has an even number of elements, $m(A)$ is the average of the middle two elements). Determine the average of $m(A)$, when A is taken over all nonempty subsets of S.

3. [**4**] Farmer John has 5 cows, 4 pigs, and 7 horses. How many ways can he pair up the animals so that every pair consists of animals of different species? (Assume that all animals are distinguishable from each other.)

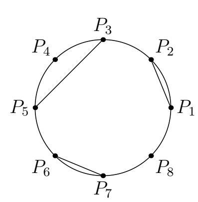
4. [**4**] Kermit the frog enjoys hopping around the infinite square grid in his backyard. It takes him 1 Joule of energy to hop one step north or one step south, and 1 Joule of energy to hop one step east or one step west. He wakes up one morning on the grid with 100 Joules of energy, and hops till he falls asleep with 0 energy. How many different places could he have gone to sleep?

5. [**5**] Let S be the smallest subset of the integers with the property that $0 \in S$ and for any $x \in S$, we have $3x \in S$ and $3x + 1 \in S$. Determine the number of non-negative integers in S less than 2008.

6. [**5**] A *Sudoku matrix* is defined as a 9×9 array with entries from $\{1, 2, \ldots, 9\}$ and with the constraint that each row, each column, and each of the nine 3×3 boxes that tile the array contains each digit from 1 to 9 exactly once. A Sudoku matrix is chosen at random (so that every Sudoku matrix has equal probability of being chosen). We know two of squares in this matrix, as shown. What is the probability that the square marked by ? contains the digit 3?



7. [**6**] Let $P_1, P_2, \ldots, P_8$ be 8 distinct points on a circle. Determine the number of possible configurations made by drawing a set of line segments connecting pairs of these 8 points, such that: (1) each P_i is the endpoint of at most one segment and (2) two no segments intersect. (The configuration with no edges drawn is allowed. An example of a valid configuration is shown below.)



8. [**6**] Determine the number of ways to select a sequence of 8 sets $A_1, A_2, \ldots, A_8$, such that each is a subset (possibly empty) of $\{1, 2\}$, and A_m contains A_n if m divides n.

9. [**7**] On an infinite chessboard (whose squares are labeled by (x, y), where x and y range over all integers), a king is placed at $(0, 0)$. On each turn, it has probability of 0.1 of moving to each of the four edge-neighboring squares, and a probability of 0.05 of moving to each of the four diagonally-neighboring squares, and a probability of 0.4 of not moving. After 2008 turns, determine the probability that the king is on a square with both coordinates even. An exact answer is required.

10. [**7**] Determine the number of 8-tuples of nonnegative integers $(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)$ satisfying $0 \le a_k \le k$, for each $k = 1, 2, 3, 4$, and $a_1 + a_2 + a_3 + a_4 + 2b_1 + 3b_2 + 4b_3 + 5b_4 = 19$.

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Individual Round: Calculus Test

1. [**3**] Let $f(x) = 1 + x + x^2 + \cdots + x^{100}$. Find $f'(1)$.
2. [**3**] Let ℓ be the line through $(0,0)$ and tangent to the curve $y = x^3 + x + 16$. Find the slope of ℓ.
3. [**4**] Find all $y > 1$ satisfying $\displaystyle\int_1^y x \ln x \, dx = \frac{1}{4}$.
4. [**4**] Let a, b be constants such that $\displaystyle\lim_{x\to 1} \frac{(\ln(2-x))^2}{x^2 + ax + b} = 1$. Determine the pair (a, b).
5. [**4**] Let $f(x) = \sin^6\left(\frac{x}{4}\right) + \cos^6\left(\frac{x}{4}\right)$ for all real numbers x. Determine $f^{(2008)}(0)$ (i.e., f differentiated 2008 times and then evaluated at $x = 0$).
6. [**5**] Determine the value of $\displaystyle\lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k}^{-1}$.
7. [**5**] Find p so that $\lim_{x\to\infty} x^p \left(\sqrt[3]{x+1} + \sqrt[3]{x-1} - 2\sqrt[3]{x}\right)$ is some non-zero real number.
8. [**7**] Let $T = \displaystyle\int_0^{\ln 2} \frac{2e^{3x} + e^{2x} - 1}{e^{3x} + e^{2x} - e^x + 1} \, dx$. Evaluate e^T.
9. [**7**] Evaluate the limit $\displaystyle\lim_{n\to\infty} n^{-\frac{1}{2}\left(1+\frac{1}{n}\right)} \left(1^1 \cdot 2^2 \cdot \cdots \cdot n^n\right)^{\frac{1}{n^2}}$.
10. [**8**] Evaluate the integral $\displaystyle\int_0^1 \ln x \ln(1-x) \, dx$.

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Individual Round: General Test, Part 1

1. [**2**] Let $ABCD$ be a unit square (that is, the labels A, B, C, D appear in that order around the square). Let X be a point outside of the square such that the distance from X to AC is equal to the distance from X to BD, and also that $AX = \frac{\sqrt{2}}{2}$. Determine the value of CX^2.

2. [**3**] Find the smallest positive integer n such that $107n$ has the same last two digits as n.

3. [**3**] There are 5 dogs, 4 cats, and 7 bowls of milk at an animal gathering. Dogs and cats are distinguishable, but all bowls of milk are the same. In how many ways can every dog and cat be paired with either a member of the other species or a bowl of milk such that all the bowls of milk are taken?

4. [**3**] Positive real numbers x, y satisfy the equations $x^2 + y^2 = 1$ and $x^4 + y^4 = \frac{17}{18}$. Find xy.

5. [**4**] The function f satisfies

$$f(x) + f(2x + y) + 5xy = f(3x - y) + 2x^2 + 1$$

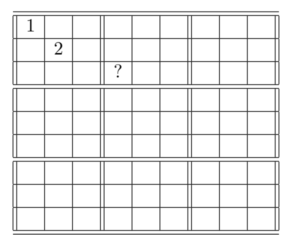
for all real numbers x, y. Determine the value of $f(10)$.

6. [**4**] In a triangle ABC, take point D on BC such that $DB = 14, DA = 13, DC = 4$, and the circumcircle of ADB is congruent to the circumcircle of ADC. What is the area of triangle ABC?

7. [**5**] The equation $x^3 - 9x^2 + 8x + 2 = 0$ has three real roots p, q, r. Find $\frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2}$.

8. [**5**] Let S be the smallest subset of the integers with the property that $0 \in S$ and for any $x \in S$, we have $3x \in S$ and $3x + 1 \in S$. Determine the number of positive integers in S less than 2008.

9. [**5**] A *Sudoku matrix* is defined as a 9×9 array with entries from $\{1, 2, \ldots, 9\}$ and with the constraint that each row, each column, and each of the nine 3×3 boxes that tile the array contains each digit from 1 to 9 exactly once. A Sudoku matrix is chosen at random (so that every Sudoku matrix has equal probability of being chosen). We know two of squares in this matrix, as shown. What is the probability that the square marked by ? contains the digit 3?



10. [**6**] Let ABC be an equilateral triangle with side length 2, and let Γ be a circle with radius $\frac{1}{2}$ centered at the center of the equilateral triangle. Determine the length of the shortest path that starts somewhere on Γ, visits all three sides of ABC, and ends somewhere on Γ (not necessarily at the starting point). Express your answer in the form of $\sqrt{p} - q$, where p and q are rational numbers written as reduced fractions.

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Individual Round: General Test, Part 2

1. [**2**] Four students from Harvard, one of them named Jack, and five students from MIT, one of them named Jill, are going to see a Boston Celtics game. However, they found out that only 5 tickets remain, so 4 of them must go back. Suppose that at least one student from each school must go see the game, and at least one of Jack and Jill must go see the game, how many ways are there of choosing which 5 people can see the game?

2. [**2**] Let ABC be an equilateral triangle. Let Ω be a circle inscribed in ABC and let ω be a circle tangent externally to Ω as well as to sides AB and AC. Determine the ratio of the radius of Ω to the radius of ω.

3. [**3**] A $3 \times 3 \times 3$ cube composed of 27 unit cubes rests on a horizontal plane. Determine the number of ways of selecting two distinct unit cubes (order is irrelevant) from a $3 \times 3 \times 1$ block with the property that the line joining the centers of the two cubes makes a $45°$ angle with the horizontal plane.

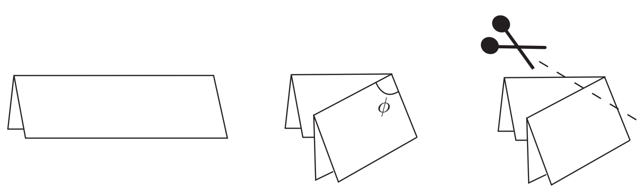
4. [**3**] Suppose that a, b, c, d are real numbers satisfying $a \geq b \geq c \geq d \geq 0$, $a^2 + d^2 = 1$, $b^2 + c^2 = 1$, and $ac + bd = 1/3$. Find the value of $ab - cd$.

5. [**4**] Kermit the frog enjoys hopping around the infinite square grid in his backyard. It takes him 1 Joule of energy to hop one step north or one step south, and 1 Joule of energy to hop one step east or one step west. He wakes up one morning on the grid with 100 Joules of energy, and hops till he falls asleep with 0 energy. How many different places could he have gone to sleep?

6. [**4**] Determine all real numbers a such that the inequality $|x^2 + 2ax + 3a| \leq 2$ has exactly one solution in x.

7. [**5**] A *root of unity* is a complex number that is a solution to $z^n = 1$ for some positive integer n. Determine the number of roots of unity that are also roots of $z^2 + az + b = 0$ for some integers a and b.

8. [**5**] A piece of paper is folded in half. A second fold is made such that the angle marked below has measure ϕ ($0° < \phi < 90°$), and a cut is made as shown below.



When the piece of paper is unfolded, the resulting hole is a polygon. Let O be one of its vertices. Suppose that all the other vertices of the hole lie on a circle centered at O, and also that $\angle XOY = 144°$, where X and Y are the the vertices of the hole adjacent to O. Find the value(s) of ϕ (in degrees).

9. [**6**] Let ABC be a triangle, and I its incenter. Let the incircle of ABC touch side BC at D, and let lines BI and CI meet the circle with diameter AI at points P and Q, respectively. Given $BI = 6, CI = 5, DI = 3$, determine the value of $(DP/DQ)^2$.

10. [**6**] Determine the number of 8-tuples of nonnegative integers $(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)$ satisfying $0 \leq a_k \leq k$, for each $k = 1, 2, 3, 4$, and $a_1 + a_2 + a_3 + a_4 + 2b_1 + 3b_2 + 4b_3 + 5b_4 = 19$.

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Guts Round

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1. [**5**] Determine all pairs (a, b) of real numbers such that $10, a, b, ab$ is an arithmetic progression.
2. [**5**] Given right triangle ABC, with $AB = 4, BC = 3$, and $CA = 5$. Circle ω passes through A and is tangent to BC at C. What is the radius of ω?
3. [**5**] How many ways can you color the squares of a 2×2008 grid in 3 colors such that no two squares of the same color share an edge?

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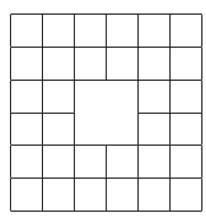
4. [**6**] Find the real solution(s) to the equation $(x + y)^2 = (x + 1)(y - 1)$.
5. [**6**] A Vandal and a Moderator are editing a Wikipedia article. The article originally is error-free. Each day, the Vandal introduces one new error into the Wikipedia article. At the end of the day, the moderator checks the article and has a $2/3$ chance of catching each individual error still in the article. After 3 days, what is the probability that the article is error-free?
6. [**6**] Determine the number of non-degenerate rectangles whose edges lie completely on the grid lines of the following figure.



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7. [**6**] Given that $x + \sin y = 2008$ and $x + 2008 \cos y = 2007$, where $0 \le y \le \pi/2$, find the value of $x + y$.
8. [**6**] Trodgor the dragon is burning down a village consisting of 90 cottages. At time $t = 0$ an angry peasant arises from each cottage, and every 8 minutes (480 seconds) thereafter another angry peasant spontaneously generates from each non-burned cottage. It takes Trodgor 5 seconds to either burn a peasant or to burn a cottage, but Trodgor cannot begin burning cottages until all the peasants around him have been burned. How many **seconds** does it take Trodgor to burn down the entire village?
9. [**6**] Consider a circular cone with vertex V, and let ABC be a triangle inscribed in the base of the cone, such that AB is a diameter and $AC = BC$. Let L be a point on BV such that the volume of the cone is 4 times the volume of the tetrahedron $ABCL$. Find the value of BL/LV.

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10. [**7**] Find the number of subsets S of $\{1, 2, \ldots 63\}$ the sum of whose elements is 2008.

11. [**7**] Let $f(r) = \sum_{j=2}^{2008} \frac{1}{j^r} = \frac{1}{2^r} + \frac{1}{3^r} + \cdots + \frac{1}{2008^r}$. Find $\sum_{k=2}^{\infty} f(k)$.

12. [**7**] Suppose we have an (infinite) cone $\mathcal{C}$ with apex A and a plane π. The intersection of π and $\mathcal{C}$ is an ellipse $\mathcal{E}$ with major axis BC, such that B is closer to A than C, and $BC = 4$, $AC = 5$, $AB = 3$. Suppose we inscribe a sphere in each part of $\mathcal{C}$ cut up by $\mathcal{E}$ with both spheres tangent to $\mathcal{E}$. What is the ratio of the radii of the spheres (smaller to larger)?

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13. [**8**] Let $P(x)$ be a polynomial with degree 2008 and leading coefficient 1 such that

$$P(0) = 2007, P(1) = 2006, P(2) = 2005, \ldots, P(2007) = 0.$$

Determine the value of $P(2008)$. You may use factorials in your answer.

14. [**8**] Evaluate the infinite sum $\sum_{n=1}^{\infty} \frac{n}{n^4+4}$.

15. [**8**] In a game show, Bob is faced with 7 doors, 2 of which hide prizes. After he chooses a door, the host opens three other doors, of which one is hiding a prize. Bob chooses to switch to another door. What is the probability that his new door is hiding a prize?

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16. [**9**] Point A lies at $(0, 4)$ and point B lies at $(3, 8)$. Find the x-coordinate of the point X on the x-axis maximizing $\angle AXB$.

17. [**9**] Solve the equation

$$\sqrt{x + \sqrt{4x + \sqrt{16x + \sqrt{\ldots + \sqrt{4^{2008}x + 3}}}}} - \sqrt{x} = 1.$$

Express your answer as a reduced fraction with the numerator and denominator written in their prime factorization.

18. [**9**] Let ABC be a right triangle with $\angle A = 90°$. Let D be the midpoint of AB and let E be a point on segment AC such that $AD = AE$. Let BE meet CD at F. If $\angle BFC = 135°$, determine BC/AB.

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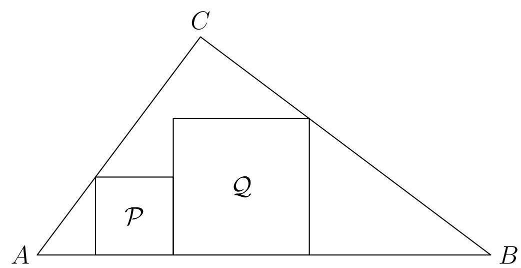
19. [**10**] Let $ABCD$ be a regular tetrahedron, and let O be the centroid of triangle BCD. Consider the point P on AO such that P minimizes $PA + 2(PB + PC + PD)$. Find $\sin \angle PBO$.

20. [**10**] For how many ordered triples (a, b, c) of positive integers are the equations $abc + 9 = ab + bc + ca$ and $a + b + c = 10$ satisfied?

21. [**10**] Let ABC be a triangle with $AB = 5$, $BC = 4$ and $AC = 3$. Let $\mathcal{P}$ and $\mathcal{Q}$ be squares inside ABC with disjoint interiors such that they both have one side lying on AB. Also, the two squares each have an edge lying on a common line perpendicular to AB, and $\mathcal{P}$ has one vertex on AC and $\mathcal{Q}$ has one vertex on BC. Determine the minimum value of the sum of the areas of the two squares.



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22. [**10**] For a positive integer n, let $\theta(n)$ denote the number of integers $0 \le x < 2010$ such that $x^2 - n$ is divisible by 2010. Determine the remainder when $\sum_{n=0}^{2009} n \cdot \theta(n)$ is divided by 2010.

23. [**10**] Two mathematicians, Kelly and Jason, play a cooperative game. The computer selects some secret positive integer $n < 60$ (both Kelly and Jason know that $n < 60$, but that they don't know what the value of n is). The computer tells Kelly the unit digit of n, and it tells Jason the number of divisors of n. Then, Kelly and Jason have the following dialogue:

    Kelly: I don't know what n is, and I'm sure that you don't know either. However, I know that n is divisible by at least two different primes.

    Jason: Oh, then I know what the value of n is.

    Kelly: Now I also know what n is.

    Assuming that both Kelly and Jason speak truthfully and to the best of their knowledge, what are all the possible values of n?

24. [**10**] Suppose that ABC is an isosceles triangle with $AB = AC$. Let P be the point on side AC so that $AP = 2CP$. Given that $BP = 1$, determine the maximum possible area of ABC.

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25. [**12**] Alice and the Cheshire Cat play a game. At each step, Alice either (1) gives the cat a penny, which causes the cat to change the number of (magic) beans that Alice has from n to $5n$ or (2) gives the cat a nickel, which causes the cat to give Alice another bean. Alice wins (and the cat disappears) as soon as the number of beans Alice has is greater than 2008 and has last two digits 42. What is the minimum number of cents Alice can spend to win the game, assuming she starts with 0 beans?

26. [**12**] Let $\mathcal{P}$ be a parabola, and let V_1 and F_1 be its vertex and focus, respectively. Let A and B be points on $\mathcal{P}$ so that $\angle AV_1B = 90^\circ$. Let $\mathcal{Q}$ be the locus of the midpoint of AB. It turns out that $\mathcal{Q}$ is also a parabola, and let V_2 and F_2 denote its vertex and focus, respectively. Determine the ratio F_1F_2/V_1V_2.

27. [**12**] Cyclic pentagon $ABCDE$ has a right angle $\angle ABC = 90^\circ$ and side lengths $AB = 15$ and $BC = 20$. Supposing that $AB = DE = EA$, find CD.

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28. [**15**] Let P be a polyhedron where every face is a regular polygon, and every edge has length 1. Each vertex of P is incident to two regular hexagons and one square. Choose a vertex V of the polyhedron. Find the volume of the set of all points contained in P that are closer to V than to any other vertex.

29. [**15**] Let (x, y) be a pair of real numbers satisfying

$$56x + 33y = \frac{-y}{x^2 + y^2}, \qquad \text{and} \qquad 33x - 56y = \frac{x}{x^2 + y^2}.$$

Determine the value of $|x| + |y|$.

30. [**15**] Triangle ABC obeys $AB = 2AC$ and $\angle BAC = 120^\circ$. Points P and Q lie on segment BC such that

$$AB^2 + BC \cdot CP = BC^2$$
$$3AC^2 + 2BC \cdot CQ = BC^2$$

Find $\angle PAQ$ in degrees.

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31. [**18**] Let $\mathcal{C}$ be the hyperbola $y^2 - x^2 = 1$. Given a point P_0 on the x-axis, we construct a sequence of points (P_n) on the x-axis in the following manner: let ℓ_n be the line with slope 1 passing passing through P_n, then P_{n+1} is the orthogonal projection of the point of intersection of ℓ_n and $\mathcal{C}$ onto the x-axis. (If $P_n = 0$, then the sequence simply terminates.)

    Let N be the number of starting positions P_0 on the x-axis such that $P_0 = P_{2008}$. Determine the remainder of N when divided by 2008.

32. [**18**] Cyclic pentagon $ABCDE$ has side lengths $AB = BC = 5$, $CD = DE = 12$, and $AE = 14$. Determine the radius of its circumcircle.

33. [**18**] Let a, b, c be nonzero real numbers such that $a + b + c = 0$ and $a^3 + b^3 + c^3 = a^5 + b^5 + c^5$. Find the value of $a^2 + b^2 + c^2$.

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34. **Who Wants to Be a Millionaire.** In 2000, the Clay Mathematics Institute named seven *Millennium Prize Problems*, with each carrying a prize of \$1 Million for its solution. Write down the name of ONE of the seven Clay Millennium Problems. If your submission is incorrect or misspelled, then your submission is disqualified. If another team wrote down the same Millennium Problem as you, then you get 0 points, otherwise you get 20 points.

35. **NUMB3RS.** The RSA Factoring Challenge, which ended in 2007, challenged computational mathematicians to factor extremely large numbers that were the product of two prime numbers. The largest number successfully factored in this challenge was RSA-640, which has 193 decimal digits and carried a prize of $\$20,000$. The next challenge number carried prize of $\$30,000$, and contains N decimal digits. Your task is to submit a guess for N. Only the team(s) that have the closest guess(es) receives points. If k teams all have the closest guesses, then each of them receives $\lceil \frac{20}{k} \rceil$ points.

36. **The History Channel.** Below is a list of famous mathematicians. Your task is to list a subset of them in the chronological order of their birth dates. Your submission should be a sequence of letters. If your sequence is not in the correct order, then you get 0 points. Otherwise your score will be $\min\{\max\{5(N-4), 0\}, 25\}$, where N is the number of letters in your sequence.

(A) Niels Abel (B) Arthur Cayley (C) Augustus De Morgan (D) Gustav Dirichlet (E) Leonhard Euler (F) Joseph Fourier (G) Évariste Galois (H) Carl Friedrich Gauss (I) Marie-Sophie Germain (J) Joseph Louis Lagrange (K) Pierre-Simon Laplace (L) Henri Poincaré (N) Bernhard Riemann

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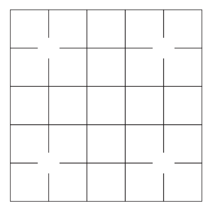
Team Round: A Division

Lattice Walks [90]

1. [**20**] Determine the number of ways of walking from $(0,0)$ to $(5,5)$ using only up and right unit steps such that the path does not pass through any of the following points: $(1,1),(1,4),(4,1),(4,4)$.



2. [**20**] Let $n>2$ be a positive integer. Prove that there are $\frac{1}{2}(n-2)(n+1)$ ways to walk from $(0,0)$ to $(n,2)$ using only up and right unit steps such that the walk never visits the line $y=x$ after it leaves the origin.

3. [**20**] Let $n>4$ be a positive integer. Determine the number of ways to walk from $(0,0)$ to $(n,2)$ using only up and right unit steps such that the path does not meet the lines $y=x$ or $y=x-n+2$ except at the start and at the end.

4. [**30**] Let $n>6$ be a positive integer. Determine the number of ways to walk from $(0,0)$ to $(n,3)$ using only up and right unit steps such that the path does not meet the lines $y=x$ or $y=x-n+3$ except at the start and at the end.

Lattice and Centroids [130]

A d-dimensional *lattice point* is a point of the form $(x_1,x_2,\ldots,x_d)$ where $x_1,x_2,\ldots,x_d$ are all integers. For a set of d-dimensional points, their *centroid* is the point found by taking the coordinate-wise average of the given set of points.

Let $f(n,d)$ denote the minimal number f such that any set of f lattice points in the d-dimensional Euclidean space contains a subset of size n whose centroid is also a lattice point.

5. [**10**] Let S be a set of 5 points in the 2-dimensional lattice. Show that we can always choose a pair of points in S whose midpoint is also a lattice point.

6. [**10**] Construct a set of 2^d d-dimensional lattice points so that for any two chosen points A,B, the line segment AB does not pass through any other lattice point.

7. [**35**] Show that for positive integers n and d,

$$(n-1)2^d+1\le f(n,d)\le (n-1)n^d+1.$$

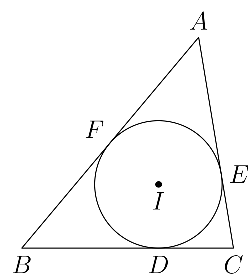
8. [**40**] Show that for positive integers n_1, n_2 and d,

$$f(n_1 n_2, d) \leq f(n_1, d) + n_1 \left(f(n_2, d) - 1 \right).$$

9. [**35**] Determine, with proof, a simple closed-form expression for $f(2^a, d)$.

Incircles [180]

In the following problems, ABC is a triangle with incenter I. Let D, E, F denote the points where the incircle of ABC touches sides BC, CA, AB, respectively.



At the end of this section you can find some terminology and theorems that may be helpful to you.

10. On the circumcircle of ABC, let A' be the midpoint of arc BC (not containing A).
    (a) [**10**] Show that A, I, A' are collinear.
    (b) [**20**] Show that A' is the circumcenter of BIC.
11. [**30**] Let lines BI and EF meet at K. Show that I, K, E, C, D are concyclic.
12. [**40**] Let K be as in the previous problem. Let M be the midpoint of BC and N the midpoint of AC. Show that K lies on line MN.
13. [**40**] Let M be the midpoint of BC, and T diametrically opposite to D on the incircle of ABC. Show that DT, AM, EF are concurrent.
14. [**40**] Let P be a point inside the incircle of ABC. Let lines DP, EP, FP meet the incircle again at D', E', F'. Show that AD', BE', CF' are concurrent.

Glossary and some possibly useful facts

- A set of points is *collinear* if they lie on a common line. A set of lines is *concurrent* if they pass through a common point. A set of points are *concyclic* if they lie on a common circle.
- Given ABC a triangle, the three angle bisectors are concurrent at the *incenter* of the triangle. The incenter is the center of the *incircle*, which is the unique circle inscribed in ABC, tangent to all three sides.

- *Ceva's theorem* states that given ABC a triangle, and points X, Y, Z on sides BC, CA, AB, respectively, the lines AX, BY, CZ are concurrent if and only if

$$\frac{BX}{XB} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

- *"Trig" Ceva* states that given ABC a triangle, and points X, Y, Z inside the triangle, the lines AX, BY, CZ are concurrent if and only if

$$\frac{\sin \angle BAX}{\sin \angle XAC} \cdot \frac{\sin \angle CBY}{\sin \angle YBA} \cdot \frac{\sin \angle ACZ}{\sin \angle ZCB} = 1.$$

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Team Round: B Division

Tropical Mathematics [95]

For real numbers x and y, let us consider the two operations $\oplus$ and $\odot$ defined by

$$x \oplus y = \min(x, y) \qquad \text{and} \qquad x \odot y = x + y.$$

We also include ∞ in our set, and it satisfies $x \oplus \infty = x$ and $x \odot \infty = \infty$ for all x. When unspecified, $\odot$ precedes $\oplus$ in the order of operations.

1. [**10**] (Distributive law) Prove that $(x \oplus y) \odot z = x \odot z \oplus y \odot z$ for all $x, y, z \in \mathbb{R} \cup \{\infty\}$.
2. [**10**] (Freshman's Dream) Let z^n denote $z \odot z \odot z \odot \cdots \odot z$ with z appearing n times. Prove that $(x \oplus y)^n = x^n \oplus y^n$ for all $x, y \in \mathbb{R} \cup \{\infty\}$ and positive integer n.
3. [**35**] By a *tropical polynomial* we mean a function of the form

   $$p(x) = a_n \odot x^n \oplus a_{n-1} \odot x^{n-1} \oplus \cdots \oplus a_1 \odot x \oplus a_0,$$

   where exponentiation is as defined in the previous problem.

   Let p be a tropical polynomial. Prove that

   $$p\left(\frac{x+y}{2}\right) \geq \frac{p(x) + p(y)}{2}$$

   for all $x, y \in \mathbb{R} \cup \{\infty\}$. (This means that all tropical polynomials are concave.)
4. [**40**] (Fundamental Theorem of Algebra) Let p be a tropical polynomial:

   $$p(x) = a_n \odot x^n \oplus a_{n-1} \odot x^{n-1} \oplus \cdots \oplus a_1 \odot x \oplus a_0, \quad a_n \neq \infty$$

   Prove that we can find $r_1, r_2, \ldots, r_n \in \mathbb{R} \cup \{\infty\}$ so that

   $$p(x) = a_n \odot (x \oplus r_1) \odot (x \oplus r_2) \odot \cdots \odot (x \oplus r_n)$$

   for all x.

Juggling [125]

A *juggling sequence* of *length* n is a sequence $j(\cdot)$ of n nonnegative integers, usually written as a string

$$j(0)j(1)\ldots j(n-1)$$

such that the mapping $f : \mathbb{Z} \to \mathbb{Z}$ defined by

$$f(t) = t + j(\bar{t})$$

is a permutation of the integers. Here $\bar{t}$ denotes the remainder of t when divided by n. In this case, we say that f is the corresponding *juggling pattern*.

For a juggling pattern f (or its corresponding juggling sequence), we say that it has b *balls* if the permutation induces b infinite orbits on the set of integers. Equivalently, b is the maximum number such that we can find a set of b integers $\{t_1, t_2, \ldots, t_b\}$ so that the sets $\{t_i, f(t_i), f(f(t_i)), f(f(f(t_i))), \ldots\}$ are all infinite and mutually disjoint (i.e. non-overlapping) for $i = 1, 2, \ldots, b$. (This definition will become clear in a second.)

Now is probably a good time to pause and think about what all this has to do with juggling. Imagine that we are juggling a number of balls, and at time t, we toss a ball from our hand up to a height $j(\bar{t})$. This ball stays up in the air for $j(\bar{t})$ units of time, so that it comes back to our hand at time $f(t) = t + j(\bar{t})$. Then, the juggling pattern presents a simplified model of how balls are juggled (for instance, we ignore information such as which hand we use to toss the ball). A throw height of 0 (i.e., $j(\bar{t}) = 0$ and $f(t) = t$) represents that no thrown takes place at time t, which could correspond to an empty hand. Then, b is simply the minimum number of balls needed to carry out the juggling.

The following graphical representation may be helpful to you. On a horizontal line, an curve is drawn from t to $f(t)$. For instance, the following diagram depicts the juggling sequence 441 (or the juggling sequences 414 and 144). Then b is simply the number of contiguous "paths" drawn, which is 3 in this case.

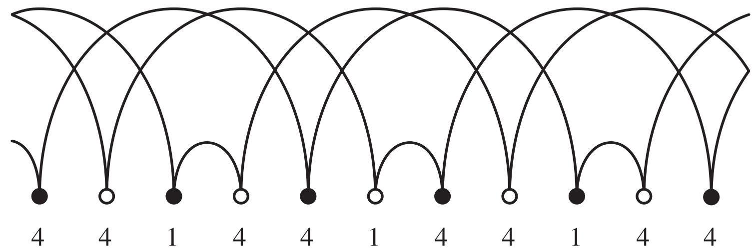


Figure 1: Juggling diagram of 441.

5. [**10**] Prove that 572 is not a juggling sequence.

6. [**40**] Suppose that $j(0)j(1)\cdots j(n-1)$ is a valid juggling sequence. For $i = 0, 1, \ldots, n-1$, Let a_i denote the remainder of $j(i) + i$ when divided by n. Prove that $(a_0, a_1, \ldots, a_{n-1})$ is a permutation of $(0, 1, \ldots, n-1)$.

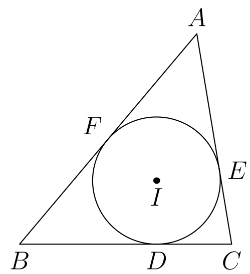
7. [**30**] Determine the number of juggling sequences of length n with exactly 1 ball.

8. [**40**] Prove that the number of balls b in a juggling sequence $j(0)j(1)\cdots j(n-1)$ is simply the average
$$b = \frac{j(0) + j(1) + \cdots + j(n-1)}{n}.$$

9. [**5**] Show that the converse of the previous statement is false by providing a non-juggling sequence $j(0)j(1)j(2)$ of length 3 where the average $\frac{1}{3}(j(0) + j(1) + j(2))$ is an integer. Show that your example works.

Incircles [180]

In the following problems, ABC is a triangle with incenter I. Let D, E, F denote the points where the incircle of ABC touches sides BC, CA, AB, respectively.



At the end of this section you can find some terminology and theorems that may be helpful to you.

10. [**15**] Let a, b, c denote the side lengths of BC, CA, AB. Find the lengths of AE, BF, CD in terms of a, b, c.

11. [**15**] Show that lines AD, BE, CF pass through a common point.

12. [**35**] Show that the incenter of triangle AEF lies on the incircle of ABC.

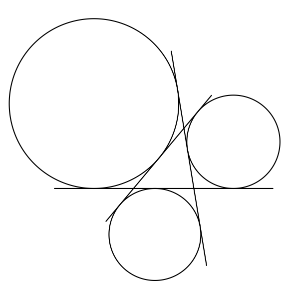
13. [**35**] Let A_1, B_1, C_1 be the incenters of triangle AEF, BDF, CDE, respectively. Show that A_1D, B_1E, C_1F all pass through the orthocenter of $A_1B_1C_1$.

14. [**40**] Let X be the point on side BC such that $BX = CD$. Show that the excircle ABC opposite of vertex A touches segment BC at X.

15. [**40**] Let X be as in the previous problem. Let T be the point diametrically opposite to D on on the incircle of ABC . Show that A, T, X are collinear.

Glossary and some possibly useful facts

- A set of points is *collinear* if they lie on a common line. A set of lines is *concurrent* if they pass through a common point.
- Given ABC a triangle, the three angle bisectors are concurrent at the *incenter* of the triangle. The incenter is the center of the *incircle*, which is the unique circle inscribed in ABC, tangent to all three sides.
- The *excircles* of a triangle ABC are the three circles on the exterior the triangle but tangent to all three lines AB, BC, CA.



- The *orthocenter* of a triangle is the point of concurrency of the three altitudes.
- *Ceva's theorem* states that given ABC a triangle, and points X, Y, Z on sides BC, CA, AB, respectively, the lines AX, BY, CZ are concurrent if and only if

$$\frac{BX}{XB} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$