

Analysis and Differential Equations

Individual

(Please select 5 problems to solve)

1.

- a) Let $x_k, k = 1, \dots, n$ be real numbers from the interval $(0, \pi)$

and define $x = \frac{\sum_{i=1}^n x_i}{n}$. Show that

$$\prod_{k=1}^n \frac{\sin x_k}{x_k} \leq \left(\frac{\sin x}{x} \right)^n.$$

- b) From

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2},$$

calculate the integral $\int_0^\infty \sin(x^2) dx$.

2. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be any function. Prove that the set of points x in \mathbb{R} where f is continuous is a countable intersection of open sets.

3. Consider the equation $\dot{x} = -x + f(t, x)$, where $|f(t, x)| \leq \phi(t)|x|$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$, $\int_0^\infty \phi(t) dt < \infty$. Prove that every solution approaches zero as $t \rightarrow \infty$.

4. Find a harmonic function f on the right half-plane such that when approaching any point in the positive half of the y -axis, the function has limit 1, while when approaching any point in the negative half of the y -axis, the function has limit -1 .

5. Let $K(x, y) \in C([0, 1] \times [0, 1])$. For all $f \in C[0, 1]$, the space of continuous functions on $[0, 1]$, define a function

$$Tf(x) = \int_0^1 K(x, y)f(y)dy$$

Prove that $Tf \in C([0, 1])$. Moreover $\Omega = \{Tf \mid \|f\|_{sup} \leq 1\}$ is precompact in $C([0, 1])$, i.e. every sequence in Ω has a converging subsequence, here $\|f\|_{sup} = \sup\{|f(x)| \mid x \in [0, 1]\}$.

6. Prove the Poisson summation formula:

$$\sum_{n=-\infty}^{\infty} f(x + 2\pi n) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx}$$

Geometry and Topology

Individual

(Please select 5 problems to solve)

1. Let $D^* = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 < 1\}$ be the punctured unit disc in the Euclidean plane. Let g be the complete Riemannian metric on D^* with constant curvature -1 . Find the distance under the metric between the points $(e^{-2\pi}, 0)$ and $(-e^{-\pi}, 0)$.
2. Show that every closed hypersurface in \mathbb{R}^n has a point at which the second fundamental form is positive definite.
3. Prove that the real projective space $\mathbb{R}P^n$ is orientable if and only if n is odd.
4. Suppose $\pi : M_1 \rightarrow M_2$ is a C^∞ map of one connected differentiable manifold to another. And suppose for each $p \in M_1$, the differential $\pi_* : T_p M_1 \rightarrow T_{\pi(p)} M_2$ is a vector space isomorphism.
- (a). Show that if M_1 is compact, then π is a covering space projection.
- (b). Given an example where M_2 is compact but $\pi : M_1 \rightarrow M_2$ is not a covering space (but has the π_* isomorphism property).
5. Let Σ_g be the closed orientable surface of genus g . Show that if $g > 1$, then Σ_g is a covering space of Σ_2 .
6. Let M be a smooth 4-dimensional manifold. A symplectic form is a closed 2-form ω on M such that $\omega \wedge \omega$ is a nowhere vanishing 4-form.
- (a). Construct a symplectic form on \mathbb{R}^4 .
- (b). Show that there are no symplectic forms on S^4 .

Algebra, Number Theory and Combinatorics

Individual

(Please select 5 problems to solve)

1. Let V be a finite dimensional complex vector space. Let A, B be two linear endomorphisms of V satisfying $AB - BA = B$. Prove that there is a common eigenvector for A and B .
2. Let $M_2(\mathbb{R})$ be the ring of 2×2 matrices with real entries. Its group of multiplicative units is $GL_2(\mathbb{R})$, consisting of invertible matrices in $M_2(\mathbb{R})$.
 - (a) Find an injective homomorphism from the field \mathbb{C} of complex numbers into the ring $M_2(\mathbb{R})$.
 - (b) Show that if ϕ_1 and ϕ_2 are two such homomorphisms, then there exists a $g \in GL_2(\mathbb{R})$ such that $\phi_2(x) = g\phi_1(x)g^{-1}$ for all $x \in \mathbb{C}$.
 - (c) Let h be an element in $GL_2(\mathbb{R})$ whose characteristic polynomial $f(x)$ is irreducible over \mathbb{R} . Let $F \subset M_2(\mathbb{R})$ be the subring generated by h and $a \cdot I$ for all $a \in \mathbb{R}$, where I is the identity matrix. Show that F is isomorphic to \mathbb{C} .
 - (d) Let h' be any element in $GL_2(\mathbb{R})$ with the same characteristic polynomial $f(x)$ as h in (c). Show that h and h' are conjugate in $GL_2(\mathbb{R})$.
 - (e) If $f(x)$ in (c) and (d) is reducible over \mathbb{R} , will the same conclusion on h and h' hold? Give reasons.
3. Let G be a non-abelian finite group. Let $c(G)$ be the number of conjugacy classes in G . Define $\bar{c}(G) := c(G)/|G|$, ($|G| = \text{Card}(G)$).
 - (a) Prove that $\bar{c}(G) \leq \frac{5}{8}$.
 - (b) Is there a finite group H with $\bar{c}(H) = \frac{5}{8}$?
 - (c) (open ended question) Suppose that there exists a prime number p and an element $x \in G$ such that the cardinality of the conjugacy class of x is divisible by p . Find a good/sharp upper bound for $\bar{c}(G)$.
4. Let F be a splitting field over \mathbb{Q} the polynomial $x^8 - 5 \in \mathbb{Q}[x]$. Recall that F is the subfield of \mathbb{C} generated by all roots of this polynomial.

4

- (a) Find the degree $[F : \mathbb{Q}]$ of the number field F .
- (b) Determine the Galois group $\text{Gal}(F/\mathbb{Q})$.

5. Let $T \subset \mathbb{N}_{>0}$ be a finite set of positive integers. For each integer $n > 0$, define a_n to be the number of all finite sequences (t_1, \dots, t_m) with $m \leq n$, $t_i \in T$ for all $i = 1, \dots, m$ and $t_1 + \dots + t_m = n$. Prove that the infinite series

$$1 + \sum_{n \geq 1} a_n z^n \in \mathbb{C}[[z]]$$

is a *rational* function in z , and find this rational function.

6. Describe all the irreducible complex representations of the group S_4 (the symmetric group on four letters).