§1.5 行列式的计算

教学目的与要求 利用行列式性质，掌握计算行列式的值的典型方法，掌握用数学归纳法求行列式值.

例 1

$$|A| = \begin{vmatrix} 3 & 0 & 15 & 12 \\ 1 & 2 & 5 & 6 \\ 2 & 0 & -3 & 0 \\ 5 & 0 & 1 & 2 \end{vmatrix}$$

解

$$|A| = 2\begin{vmatrix} 3 & 15 & 12 \\ 2 & -3 & 0 \\ 5 & 1 & 2 \end{vmatrix} = 6\begin{vmatrix} 1 & 5 & 4 \\ 2 & -3 & 0 \\ 5 & 1 & 2 \end{vmatrix} = 6\begin{vmatrix} -9 & 3 & 0 \\ 2 & -3 & 0 \\ 5 & 1 & 2 \end{vmatrix} = 12\begin{vmatrix} -7 & 0 \\ 2 & -3 \end{vmatrix} = 3\times 21 = 252$$

例 2 设 $a_i \neq 0,\ 0 \leq i \leq n$, 求

$$|A| = \begin{vmatrix} a_0 & b_1 & \cdots & b_n \\ c_1 & a_1 & & \\ \vdots & & \ddots & \\ c_n & & & a_n \end{vmatrix}.$$

解 (法 1) 消去第 1 列

$$|A| = \begin{vmatrix} a_0 - \sum\limits_{i=1}^{n} \dfrac{b_i c_i}{a_i} & b_1 & \cdots & b_n \\ & a_1 & & \\ & & \ddots & \\ & & & a_n \end{vmatrix} = (\prod_{i=1}^{n} a_i)(a_0 - \sum_{i=1}^{n} \frac{b_i c_i}{a_i})$$

(法 2) 按第 1 列展开

$$|A| = \prod_{i=0}^{n} a_i - \sum_{i=1}^{n} b_i c_i \prod_{j=1, j\neq i}^{n} a_j$$

例 3 求

$$F_n = \begin{vmatrix} \lambda & 0 & 0 & \cdots & 0 & a_n \\ -1 & \lambda & 0 & \cdots & 0 & a_{n-1} \\ 0 & -1 & \lambda & \cdots & 0 & a_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda & a_2 \\ 0 & 0 & 0 & \cdots & -1 & \lambda + a_1 \end{vmatrix}$$

解 (法 1) 按第 1 列展开. $F_n = \lambda F_{n-1} + (-1)^{1+n}(-1)^{n-1}a_n = \lambda F_{n-1} + a_n$.
从而 $F_n = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$.

(法 2) 按第 n 列展开

$$\begin{aligned} F_n &= (-1)^{1+n}a_n \begin{vmatrix} -1 & \lambda & 0 & \cdots & 0 \\ 0 & -1 & \lambda & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda \\ 0 & 0 & 0 & \cdots & -1 \end{vmatrix} + (-1)^{2+n}a_{n-1} \begin{vmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 0 & -1 & \lambda & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda \\ 0 & 0 & 0 & \cdots & -1 \end{vmatrix} \\ &+ \cdots + (\lambda + a_1) \begin{vmatrix} \lambda & 0 & \cdots & 0 \\ -1 & \lambda & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda \\ 0 & 0 & \cdots & -1 \end{vmatrix} \\ &= (-1)^{1+n}(-1)^{n-1}a_n + (-1)^{2+n}(-1)^{n-2}\lambda a_{n-1} + \cdots + (\lambda + a_1)\lambda^{n-1} \\ &= a_n + a_{n-1}\lambda + \cdots + a_1\lambda^{n-1} + \lambda^n \end{aligned}$$

(法 3)

$$\begin{aligned} F_n &= \begin{vmatrix} \lambda & 0 & 0 & \cdots & 0 & 0 & a_n \\ -1 & \lambda & 0 & \cdots & 0 & 0 & a_{n-1} \\ 0 & -1 & \lambda & \cdots & 0 & 0 & a_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 & 0 & a_2 + (\lambda + a_1)\lambda \\ 0 & 0 & 0 & \cdots & 0 & -1 & \lambda + a_1 \end{vmatrix} = \cdots \\ &= \begin{vmatrix} 0 & & & & a_n + a_{n-1}\lambda + \cdots + a_1\lambda^{n-1} + \lambda^n \\ -1 & 0 & & & \vdots \\ & -1 & \ddots & & \vdots \\ & & \ddots & 0 & a_2 + a_1\lambda + \lambda^2 \\ & & & -1 & \lambda + a_1 \end{vmatrix} \\ &= (-1)^{n-1}(-1)^{n+1}(a_n + a_{n-1}\lambda + \cdots + a_1\lambda^{n-1} + \lambda^n) \\ &= a_n + a_{n-1}\lambda + \cdots + a_1\lambda^{n-1} + \lambda^n \end{aligned}$$

例 4

$$|A|=\begin{vmatrix} x & a & a & \cdots & a \\ a & x & a & \cdots & a \\ a & a & x & \cdots & a \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a & a & a & \cdots & x \end{vmatrix} \qquad (n>1)$$

解 (法 1) 第 2 行，第 3 行，$\cdots$, 第 n 行加到第 1 行上

$$\begin{aligned}
|A| &= \begin{vmatrix} x+(n-1)a & x+(n-1)a & x+(n-1)a & \cdots & x+(n-1)a \\ a & x & a & \cdots & a \\ a & a & x & \cdots & a \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a & a & a & \cdots & x \end{vmatrix} \\
&= [x+(n-1)a]\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a & x & a & \cdots & a \\ a & a & x & \cdots & a \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a & a & a & \cdots & x \end{vmatrix} \\
&= [x+(n-1)a]\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & x-a & 0 & \cdots & 0 \\ 0 & 0 & x-a & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & x-a \end{vmatrix} \\
&= [x+(n-1)a](x-a)^{n-1}
\end{aligned}$$

(法 2)

$$\begin{aligned}
|A| &= \begin{vmatrix} x & a & a & \cdots & a \\ a-x & x-a & 0 & \cdots & 0 \\ a-x & 0 & x-a & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a-x & 0 & 0 & \cdots & x-a \end{vmatrix} \\
&= \begin{vmatrix} x+(n-1)a & a & a & \cdots & a \\ 0 & x-a & 0 & \cdots & 0 \\ 0 & 0 & x-a & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & x-a \end{vmatrix} \\
&= [x+(n-1)a](x-a)^{n-1}
\end{aligned}$$

(法 3) 加边法.

$$
\begin{aligned}
|A| &= \begin{vmatrix} 1 & a & a & a & \cdots & a \\ 0 & x & a & a & \cdots & a \\ 0 & a & x & a & \cdots & a \\ 0 & a & a & x & \cdots & a \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & a & a & a & \cdots & x \end{vmatrix}_{n+1} \\
&= \begin{vmatrix} 1 & a & a & a & \cdots & a \\ -1 & x-a & 0 & 0 & \cdots & 0 \\ -1 & 0 & x-a & 0 & \cdots & 0 \\ -1 & 0 & 0 & x-a & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & 0 & 0 & 0 & \cdots & x-a \end{vmatrix}
\end{aligned}
$$

化为例 3 的形式.

例 5(**范德蒙** Vander Monde **行列式**)

$$
V_n = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-2} & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-2} & x_2^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-2} & x_{n-1}^{n-1} \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-2} & x_n^{n-1} \end{vmatrix}
$$

解 行消去法. 第 $n-1$ 列乘以 $-x_n$ 加到第 n 列上，第 $n-2$ 列乘以 $-x_n$ 加到第 $n-1$ 列上，一直下去， 1 列乘以 $-x_n$ 加到第 2 列上，得

$$
V_n = \begin{vmatrix} 1 & x_1-x_n & x_1^2-x_1x_n & \cdots & x_1^{n-2}-x_1^{n-3}x_n & x_1^{n-1}-x_1^{n-2}x_n \\ 1 & x_2-x_n & x_2^2-x_2x_n & \cdots & x_2^{n-2}-x_2^{n-3}x_n & x_2^{n-1}-x_2^{n-2}x_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_{n-1}-x_n & x_{n-1}^2-x_{n-1}x_n & \cdots & x_{n-1}^{n-2}-x_{n-1}^{n-3}x_n & x_{n-1}^{n-1}-x_{n-1}^{n-2}x_n \\ 1 & x_n-x_n & x_n^2-x_n^2 & \cdots & x_n^{n-2}-x_n^{n-2} & x_n^{n-1}-x_n^{n-1} \end{vmatrix}
$$

$$
= (-1)^{n+1} \begin{vmatrix} x_1-x_n & x_1^2-x_1x_n & \cdots & x_1^{n-2}-x_1^{n-3}x_n & x_1^{n-1}-x_1^{n-2}x_n \\ x_2-x_n & x_2^2-x_2x_n & \cdots & x_2^{n-2}-x_2^{n-3}x_n & x_2^{n-1}-x_2^{n-2}x_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n-1}-x_n & x_{n-1}^2-x_{n-1}x_n & \cdots & x_{n-1}^{n-2}-x_{n-1}^{n-3}x_n & x_{n-1}^{n-1}-x_{n-1}^{n-2}x_n \end{vmatrix}
$$

$$
= (-1)^{n-1}(x_1 - x_n)(x_2 - x_n)\cdots(x_{n-1} - x_n)\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-2} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-2} \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-2} \end{vmatrix}
$$

$$
= (x_n - x_1)(x_n - x_2)\cdots(x_n - x_{n-1})V_{n-1}.
$$

得递推公式 $V_n = \prod_{i=1}^{n-1}(x_n - x_i)V_{n-1}$. 所以 $V_n = \prod_{1\le j<i=\le n}(x_i - x_j)$.

注 数学归纳法类型 I: $D_n = \alpha D_{n-1} + \beta$.

例 6 $y \neq z$

$$
D_n = \begin{vmatrix} x & y & \cdots & y \\ z & x & \cdots & y \\ \cdots & \cdots & \cdots & \cdots \\ z & z & \cdots & y \\ z & z & \cdots & x \end{vmatrix} = \begin{vmatrix} x & y & \cdots & y \\ z & x & \cdots & y \\ \cdots & \cdots & \cdots & \cdots \\ z & z & \cdots & y \\ z & z & \cdots & y \end{vmatrix} + \begin{vmatrix} x & y & \cdots & 0 \\ z & x & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ z & z & \cdots & 0 \\ z & z & \cdots & x-y \end{vmatrix}
$$

$$
= (x-y)D_{n-1} + y(x-z)^{n-1}
$$

由于 y, z 有对称性, 故 $D_n = (x-z)D_{n-1} + z(x-y)^{n-1}$, 联立方程可得结果.

注 数学归纳法类型 II: $\begin{cases} D_n = aD_{n-1} + b \\ D_n = cD_{n-1} + d \end{cases}$ $(a \neq c)$

例 7

$$
D_n = \begin{vmatrix} a+b & ab & \cdots & 0 & 0 \\ 1 & a+b & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a+b & ab \\ 0 & 0 & \cdots & 1 & a+b \end{vmatrix}
$$

解 按第 1 列展开，可得 $D_n = (a+b)D_{n-1} - abD_{n-2}$.

注 数学归纳法类型 III:$D_n = pD_{n-1} + qD_{n-2}$, 设 a, b 是 $x^2 - px - q = 0$ 的根, $p = a+b$,$q = -ab$. $\begin{cases} D_n - aD_{n-1} = b(D_{n-1} - aD_{n-2}) \\ D_n - bD_{n-1} = a(D_{n-1} - bD_{n-2}) \end{cases}$, 化为类型 I, 类型 II.

练习 P_{33} 1 (1), 3(1), 4(2), 5; P_{37-38} 5, 7; P_{46-47} 14, 17, 18

选做 P_{45}, 2, 15, 19